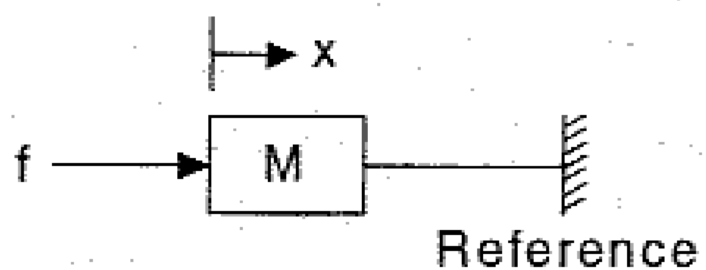
UNIT-II:

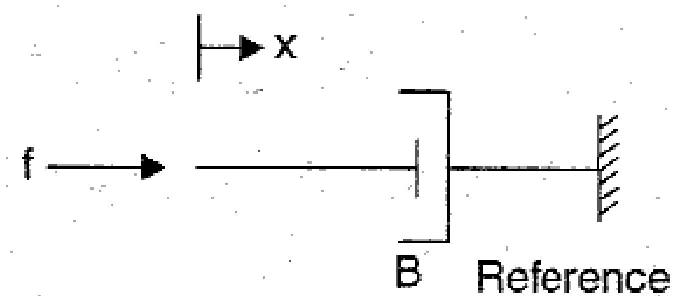
Mathematical Models: Translational and Rotational Mechanical systems, Differential equations, Analogous of Mechanical System to Electrical System using Force (Torque)-Voltage, Force (Torque)-Current, Armature and Field Controlled DC Motor, Synchro transmitter and receiver.

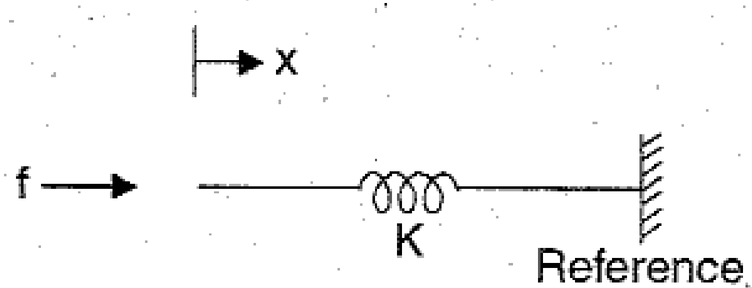
Time Response Analysis: Standard test signals, Time response of first order systems, Transient response of second order systems, Characteristic Equation, Time domain specifications, Steady state response, Steady state errors and error constants, Effects of P, PI, PD and PID controllers.

Mechanical Translatory and Rotational Systems

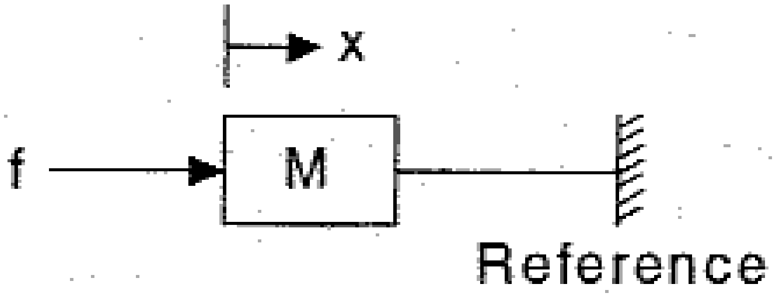
The weight of the mechanical system is represented by the element ***mass*** and it is assumed to be concentrated at the center of the body. The elastic deformation of the body can be represented by a ***spring.*** The friction existing in rotating mechanical system can be represented by the ***dash-pot.*** The dash-pot is a piston moving inside a cylinder filled with viscous fluid.







MASS ELEMENT



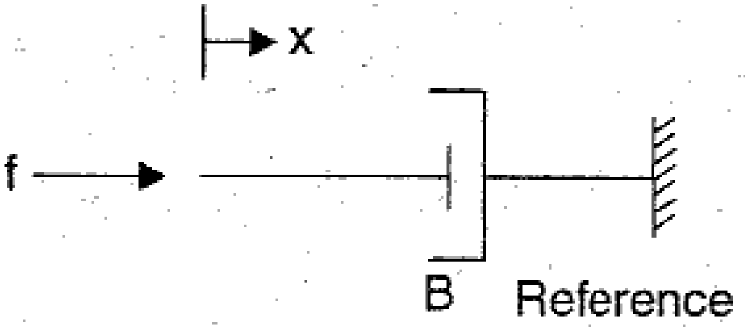
f = Applied force

f_m = Opposing force due to mass

$$f_m \propto \frac{d^2x}{dt^2} \qquad f_m = M\frac{d^2x}{dt^2}$$

By Newton's second law, $\boxed{f = f_m = M\frac{d^2x}{dt^2}}$

DASH POT ELEMENT

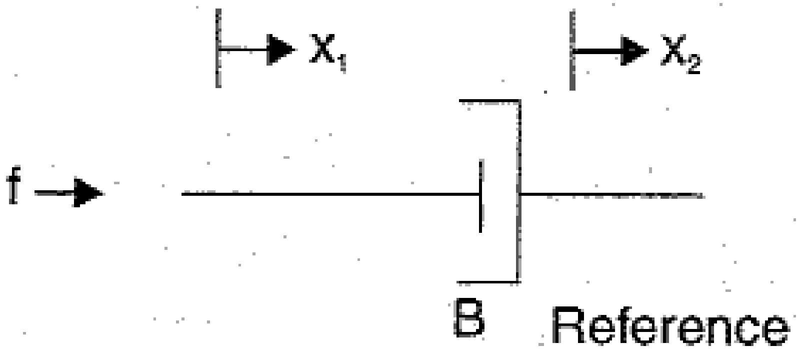


Let, f = Applied force

f_b = Opposing force due to friction

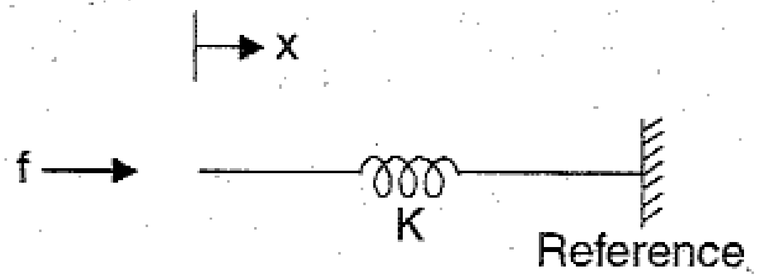
Here, $f_b \propto \frac{dx}{dt}$ or $f_b = B\frac{dx}{dt}$

By Newton's second law, $f = f_b = B\frac{dx}{dt}$



$f_b \propto \frac{d}{dt}(x_1 - x_2)$ or $f_b = B\frac{d}{dt}(x_1 - x_2)$ $f = f_b = B\frac{d}{dt}(x_1 - x_2)$

SPRING ELEMENT

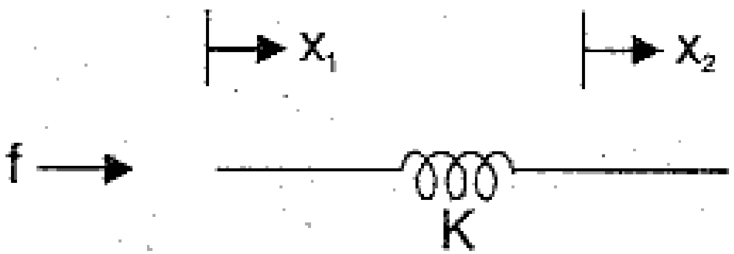


Let, f = Applied force

f_k = Opposing force due to elasticity

Here $f_k \propto x$ or $f_k = K\,x$

By Newton's second law, $\boxed{f = f_k = Kx}$



$f_k \propto (x_1 - x_2)$ or $f_k = K(x_1 - x_2)$ $\boxed{f = f_k = K(x_1 - x_2)}$

1) Mass Element $M\frac{d^2x}{dt^2}$ $Ms^2X(s)$

2) Dashpot Element $B\frac{dx}{dt}$ $BsX(s)$

$B\frac{d(x_1-x_2)}{dt}$

$Bs[X_1(s)-X_2(s)]$

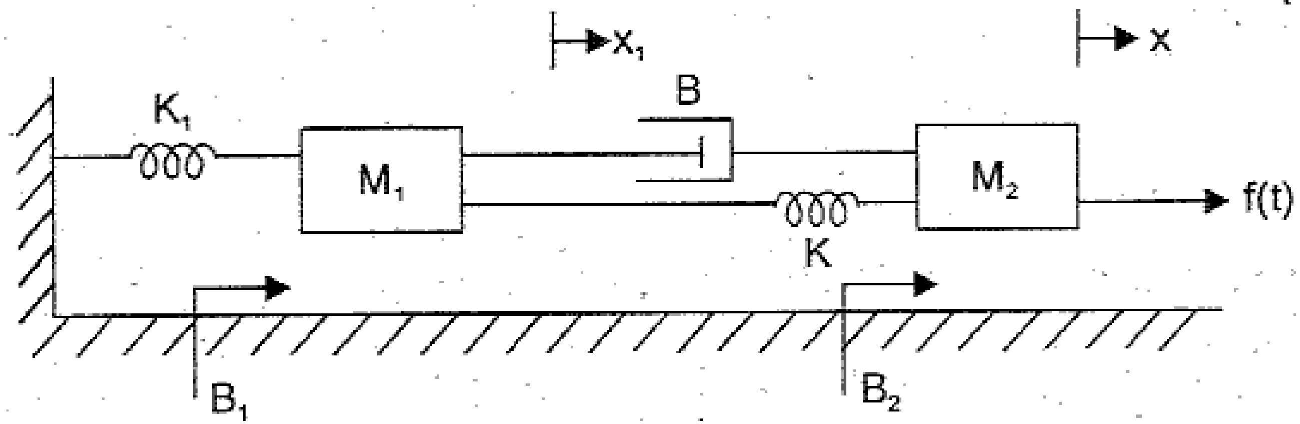
3) Spring Element Kx $KX(s)$

$k[x_1-x_2]$

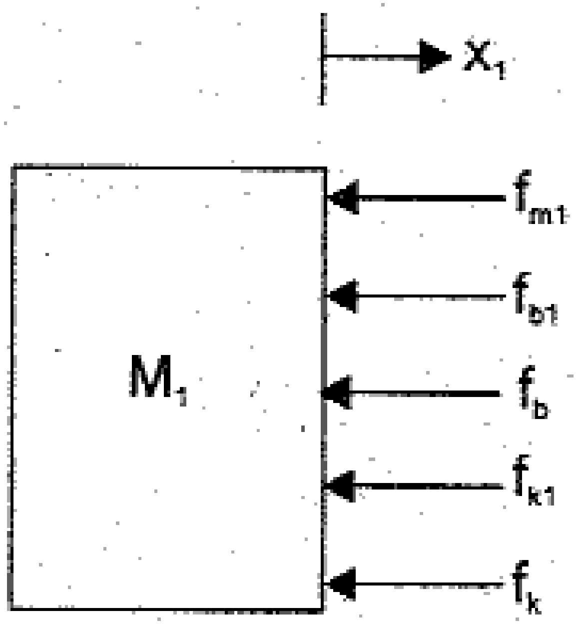
$k[X_1(s)-X_2(s)]$

Problem 1

Obtain the transfer function by writing differential equations $\frac{X(s)}{F(s)}$



The free body diagram of mass M_1 is shown in fig.



By Newton's second law,

$$f_{m1} + f_{b1} + f_b + f_{k1} + f_k = 0$$

$$\therefore M_1 \frac{d^2 x_1}{dt^2} + B_1 \frac{dx_1}{dt} + B \frac{d}{dt}(x_1 - x) + K_1 x_1 + K(x_1 - x) = 0$$

By Newton's second law,

$$f_{m1} + f_{b1} + f_b + f_{k1} + f_k = 0$$

$$\therefore M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{dx_1}{dt} + B \frac{d}{dt}(x_1 - x) + K_1 x_1 + K(x_1 - x) = 0$$

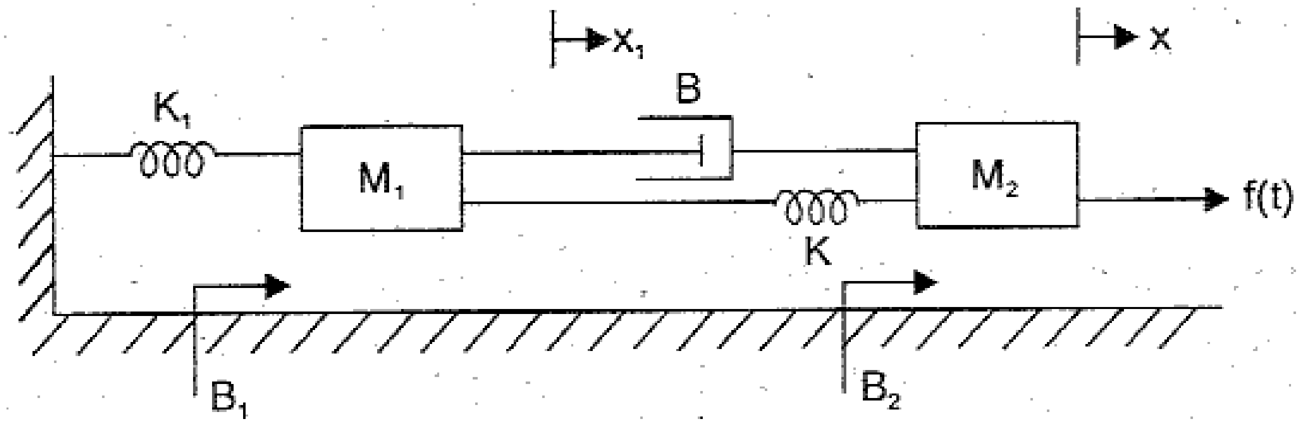
On taking Laplace transform of above equation with zero initial conditions we get,

$$M_1 s^2 X_1(s) + B_1 s X_1(s) + Bs\,[X_1(s) - X(s)] + K_1 X_1(s) + K\,[X_1(s) - X(s)] = 0$$

$$X_1(s)\,[M_1 s^2 + (B_1 + B)s + (K_1 + K)] - X(s)\,[Bs + K] = 0$$

$$X_1(s)\,[M_1 s^2 + (B_1 + B)s + (K_1 + K)] = X(s)\,[Bs + K]$$

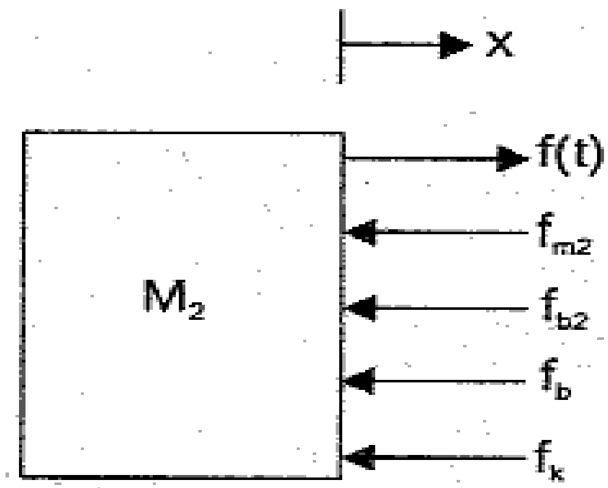
$$\therefore X_1(s) = X(s)\,\frac{Bs + K}{M_1 s^2 + (B_1 + B)\,s + (K_1 + K)}$$ ------ **Eq 1**



The free body diagram of mass M_2 is shown in fig

$$f_{m2} + f_{b2} + f_b + f_k = f(t)$$

$$M_2 \frac{d^2x}{dt^2} + B_2 \frac{dx}{dt} + B\frac{d}{dt}(x - x_1) + K(x - x_1) = f(t)$$



By Newton's second law,

$$f_{m2} + f_{b2} + f_b + f_k = f(t)$$

$$M_2 \frac{d^2x}{dt^2} + B_2 \frac{dx}{dt} + B \frac{d}{dt}(x - x_1) + K(x - x_1) = f(t)$$

On taking Laplace transform of above equation with zero initial conditions we get,

$$M_2 s^2 X(s) + B_2 s X(s) + Bs[X(s) - X_1(s)] + K[X(s) - X_1(s)] = F(s)$$

$$X(s)\,[M_2 s^2 + (B_2 + B)s + K] - X_1(s)[Bs + K] = F(s) \qquad \text{------ Eq 2}$$

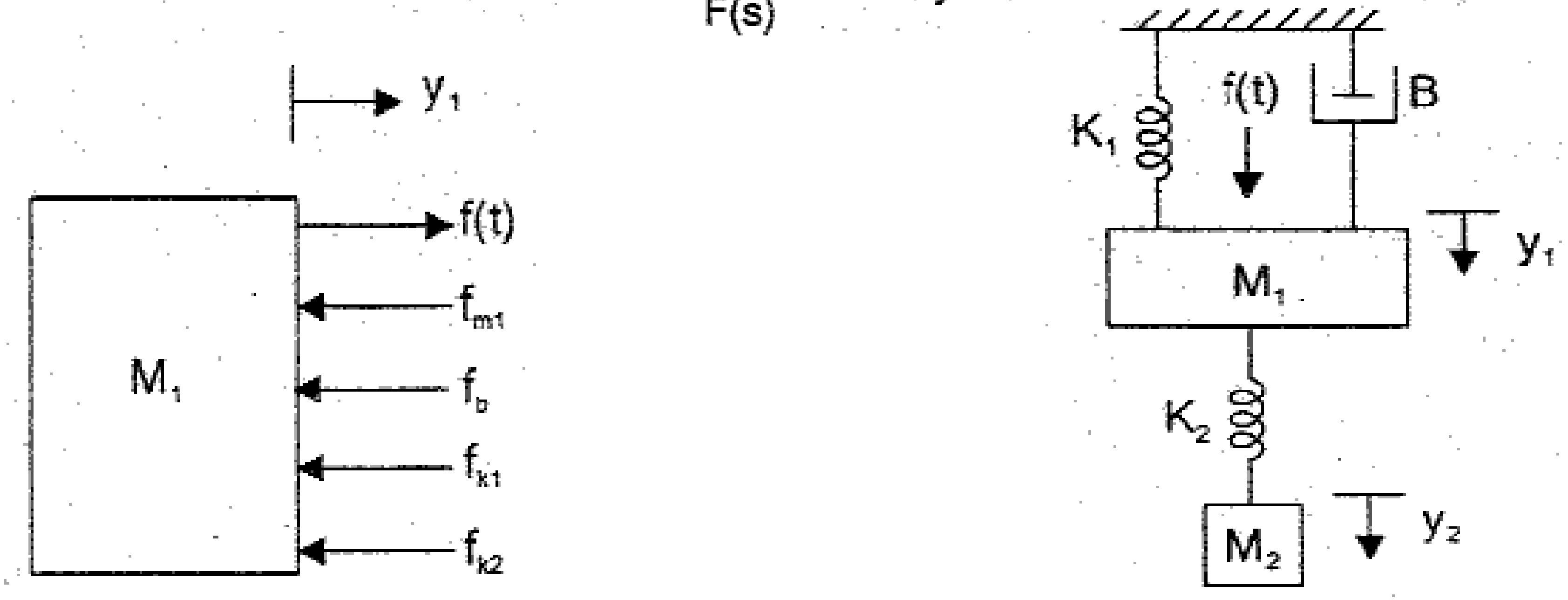
Substituting for $X_1(s)$ from equation (1) in equation (2) we get,

$$X(s)\,[M_2 s^2 + (B_2 + B)s + K] - X(s)\frac{(Bs + K)^2}{M_1 s^2 + (B_1 + B)s + (K_1 + K)} = F(s)$$

$$X(s)\left[\frac{[M_2s^2+(B_2+B)s+K]\,[M_1s^2+(B_1+B)s+(K_1+K)]-(Bs+K)^2}{M_1s^2+(B_1+B)\,s+(K_1+K)}\right]=F(s)$$

$$\therefore \frac{X(s)}{F(s)}=\frac{M_1s^2+(B_1+B)s+(K_1+K)}{[M_1s^2+(B_1+B)s+(K_1+K)]\,[M_2s^2+(B_2+B)s+K]-(Bs+K)^2}$$

Problem 2 Determine the transfer function $\frac{Y_2(s)}{F(s)}$ of the system



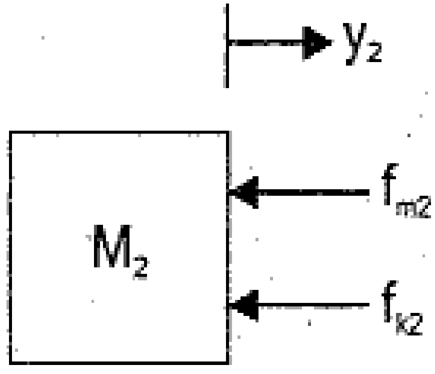
By Newton's second law, $f_{m1} + f_b + f_{k1} + f_{k2} = f(t)$

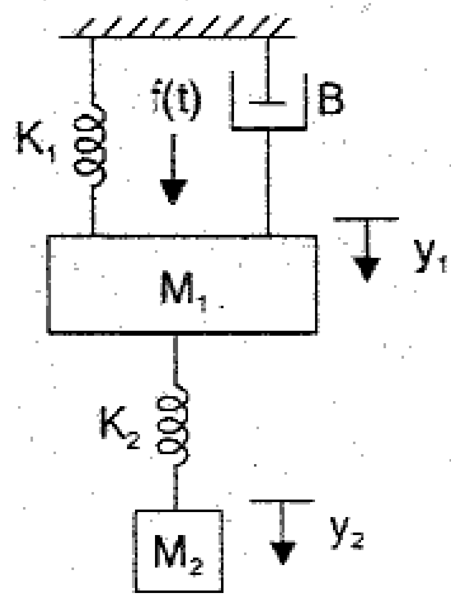
$$\therefore M_1 \frac{d^2 y_1}{dt^2} + B\frac{dy_1}{dt} + K_1 y_1 + K_2(y_1 - y_2) = f(t) \qquad .....(1)$$

On taking Laplace transform of equation (1) with zero initial condition we get,

$$M_1 s^2 Y_1(s) + BsY_1(s) + K_1 Y_1(s) + K_2[Y_1(s) - Y_2(s)] = F(s)$$

$$Y_1(s)[M_1 s^2 + Bs + (K_1 + K_2)] - Y_2(s)K_2 = F(s) \qquad \text{------ Eq 2}$$





By Newton's second law, $f_{m2} + f_{k2} = 0$

$$\therefore M_2 \frac{d^2 y_2}{dt^2} + K_2(y_2 - y_1) = 0$$

On taking Laplace transform of above equation we get,

$$M_2 s^2 Y_2(s) + K_2[Y_2(s) - Y_1(s)] = 0$$

$$Y_2(s)\ [M_2 s^2 + K_2] - Y_1(s)\ K_2 = 0$$

$$\therefore Y_1(s) = Y_2(s)\ \frac{M_2 s^2 + K_2}{K_2} \quad \text{------ Eq 3}$$

Substituting for $Y_1(s)$ from equation (3) in equation (2) we get,

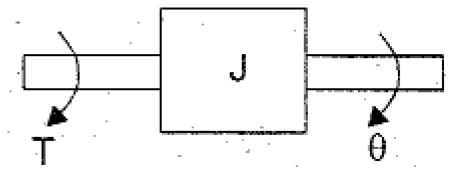
$$Y_2(s)\left[\frac{M_2s^2+K_2}{K_2}\right]\left[M_1s^2+Bs+(K_1+K_2)\right]-Y_2(s)\,K_2=F(s)$$

$$Y_2(s)\left[\frac{(M_2s^2+K_2)\,[M_1s^2+Bs+(K_1+K_2)]-K_2^2}{K_2}\right]=F(s)$$

$$\therefore \frac{Y_2(s)}{F(s)}=\frac{K_2}{\left[M_1s^2+Bs+(K_1+K_2)\right]\left[M_2s^2+K_2\right]-K_2^2}$$

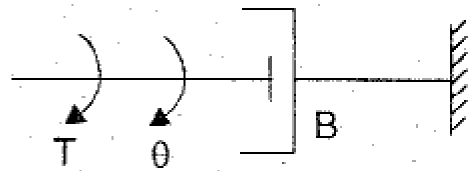
Mechanical Rotational System

Moment of Inertia

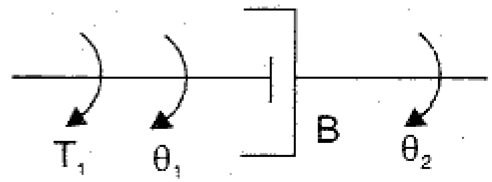


$$T = T_j = J\frac{d^2\theta}{dt^2}$$

Dash Pot with B frictional Coefficient

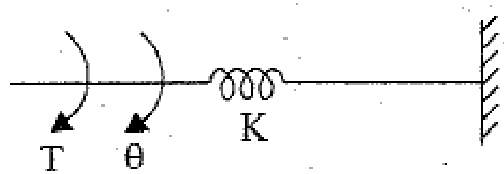


$$T = T_b = B\frac{d\theta}{dt}$$

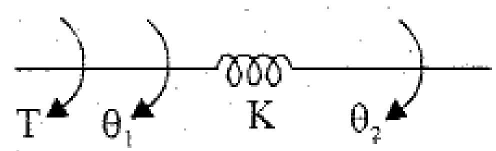


$$\therefore T = T_b = B\frac{d}{dt}(\theta_1 - \theta_2)$$

Torsional Spring



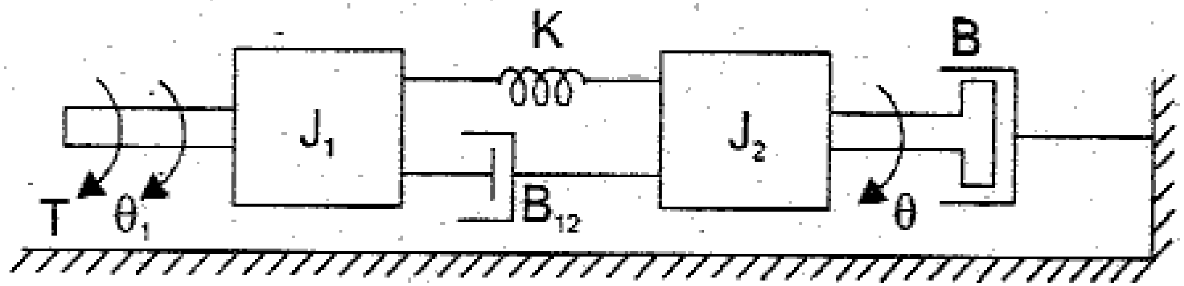
$$T = T_k = K\theta$$

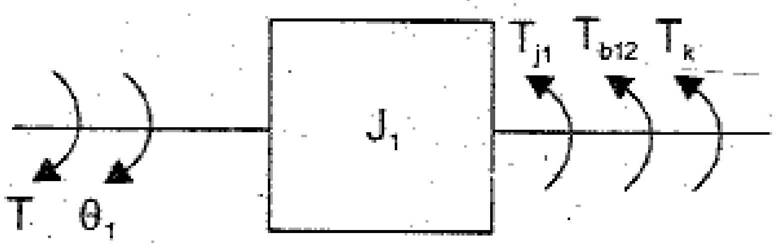


$$T = T_k = K(\theta_1 - \theta_2)$$

Problem 3

determine the transfer function $\theta(s)/T(s)$.





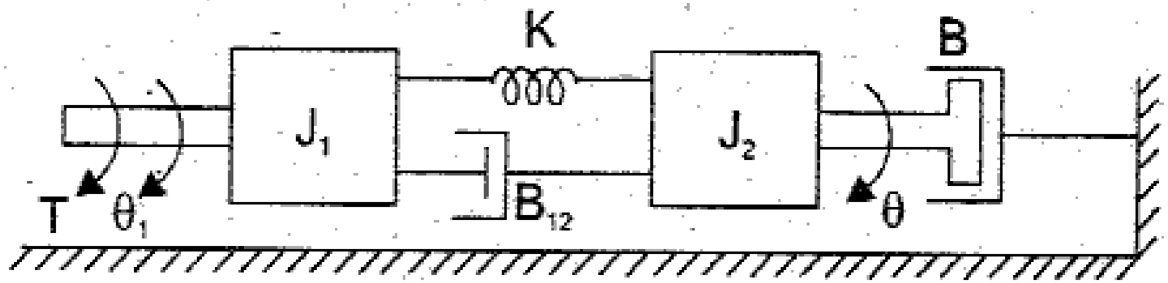
By Newton's second law, $T_{j1} + T_{b12} + T_k = T$

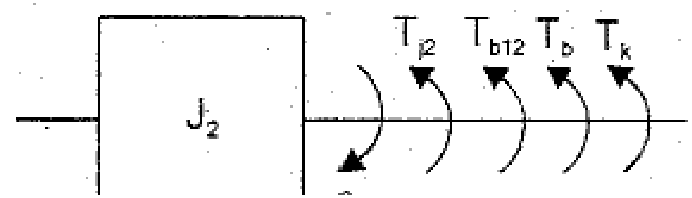
$$J_1 \frac{d^2\theta_1}{dt^2} + B_{12}\frac{d}{dt}(\theta_1 - \theta) + K(\theta_1 - \theta) = T$$

On taking Laplace transform of above equation with zero initial conditions we get,

$$J_1 s^2 \theta_1(s) + s\, B_{12}\, [\theta_1(s) - \theta(s)] + K\theta_1(s) - K\theta(s) = T(s)$$

$$\theta_1(s)\, [J_1 s^2 + sB_{12} + K] - \theta(s)\, [sB_{12} + K] = T(s)$$





By Newton's second law, $T_{j2} + T_{b12} + T_b + T_k = 0$

$$J_2 \frac{d^2\theta}{dt^2} + B_{12}\frac{d}{dt}(\theta - \theta_1) + B\frac{d\theta}{dt} + K(\theta - \theta_1) = 0$$

$$J_2 \frac{d^2\theta}{dt^2} - B_{12}\frac{d\theta_1}{dt} + \frac{d\theta}{dt}(B_{12} + B) + K\theta - K\theta_1 = 0$$

On taking Laplace transform of above equation with zero initial conditions we get,

$$J_2 s^2\theta(s) - B_{12}s\theta_1(s) + s\theta(s)\,[B_{12} + B] + K\theta(s) - K\theta_1(s) = 0$$

$$\theta(s)\,[s^2J_2 + s(B_{12} + B) + K] - \theta_1(s)\,[sB_{12} + K] = 0$$

$$\theta_1(s) = \frac{[s^2J_2 + s(B_{12} + B) + K]}{[sB_{12} + K]}\,\theta(s)$$

Substituting for $\theta_1(s)$ from equation (2) in equation (1) we get,

$$[J_1s^2 + sB_{12} + K]\ \frac{[J_2s^2 + s(B_{12} + B) + K]\ \theta(s)}{(sB_{12} + K)} - (sB_{12} + K)\ \theta(s) = T(s)$$

$$\left[\frac{(J_1s^2 + sB_{12} + K)\ [J_2s^2 + s(B_{12} + B) + K] - (sB_{12} + K)^2}{(sB_{12} + K)}\right]\theta(s) = T(s)$$

$$\therefore \frac{\theta(s)}{T(s)} = \frac{(sB_{12} + K)}{(J_1s^2 + sB_{12} + K)\ [J_2s^2 + s(B_{12} + B) + K] - (sB_{12} + K)^2}$$

$M\frac{d^2x}{dt^2}$ $B\frac{dx}{dt}$ kx

F, M, B, k

x, v $v = \frac{dx}{dt}$

$M\frac{dv}{dt}$ Bv $k\int v\,dt$

$J\frac{d^2\theta}{dt^2}$ $B\frac{d\theta}{dt}$ $k\theta$

T, J, B, k

θ, ω $\omega = \frac{d\theta}{dt}$

$J\frac{d\omega}{dt}$ $B\omega$ $k\int\omega\,d\theta$

Analogy Between Mechanical Systems and Electrical Systems

R, L, C

Voltage $e = iR$ $e = L\frac{di}{dt}$ $e = \frac{1}{C}\int i\,dt$

current $i = \frac{v}{R}$ $i = \frac{1}{L}\int v\,dt$ $i = C\frac{dv}{dt}$

$$V = \frac{dx}{dt}$$

$$M\frac{d^2x}{dt^2} \qquad M\frac{dV}{dt}$$

$$B\frac{dx}{dt} \qquad BV$$

$$kx \qquad k\int V\,dt$$

$$\omega = \frac{d\theta}{dt}$$

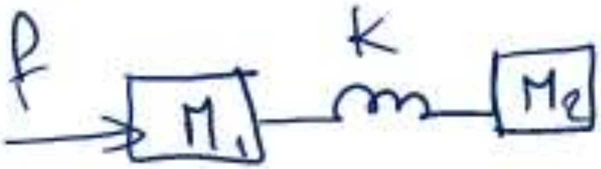
$$J\frac{d^2\theta}{dt^2} \qquad J\frac{d\omega}{dt}$$

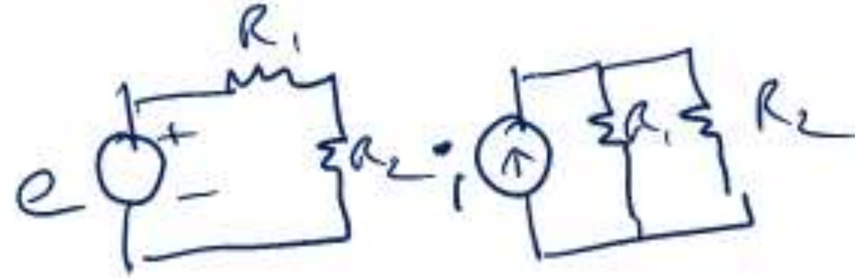
$$B\frac{d\theta}{dt} \qquad B\omega$$

$$k\theta \qquad k\int \omega\,dt$$

Analogy

Translatory	Electrical
Force	Voltage
	(or)
	Current

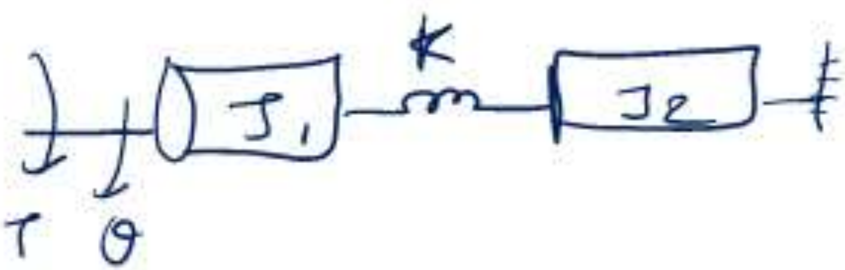


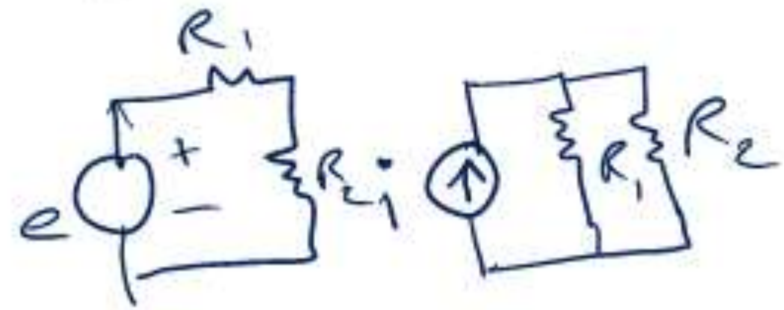


Analogy

Rotational — Electrical

Torque — Voltage (or) Current





$F, \quad M, \quad B, \quad k$

$x, \quad v = \frac{dx}{dt}$

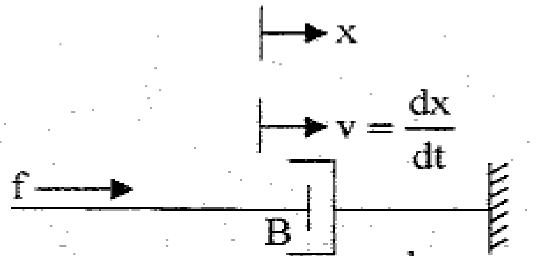
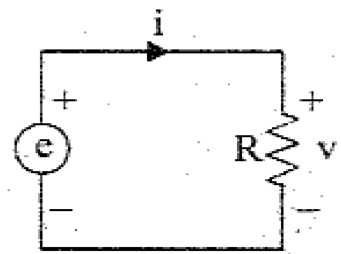
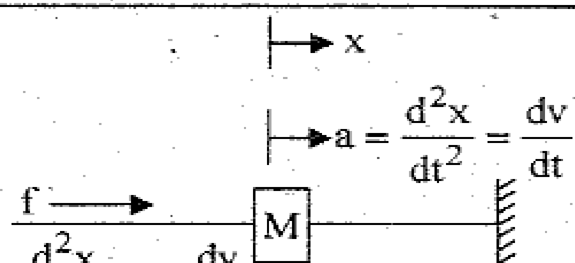
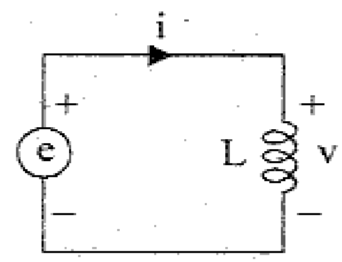
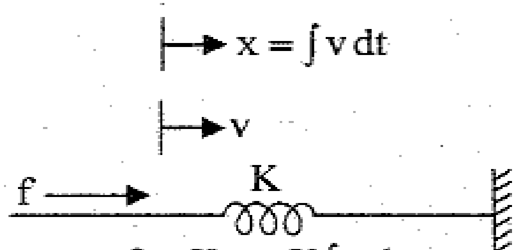
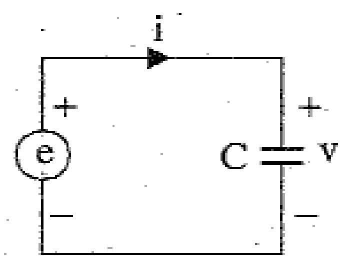
Translatory Systems

Force-Voltage Analogy

$R, \quad L, \quad C$

$e, \quad i \qquad i = \frac{dq}{dt} \qquad q$ (charge)

Analogous Elements in Force-Voltage Analogy

Mechanical system	Electrical system
Input : Force Output : Velocity	Input : Voltage source Output : Current through the element
 $v = \frac{dx}{dt}$ $f = B\frac{dx}{dt} = Bv$	 $e = v$ and $v = Ri$ $\therefore e = Ri$
 $a = \frac{d^2x}{dt^2} = \frac{dv}{dt}$ $f = M\frac{d^2x}{dt^2} = M\frac{dv}{dt}$	 $e = v$ and $v = L\frac{di}{dt}$ $\therefore e = L\frac{di}{dt}$
 $x = \int v\,dt$ $f = Kx = K\int v\,dt$	 $e = v$ and $v = \frac{1}{C}\int i\,dt$ $\therefore e = \frac{1}{C}\int i\,dt$

Analogous Elements in Force-Voltage Analogy

Item	Mechanical system	Electrical system (mesh basis system)
Independent variable (input)	Force, f	Voltage, e, v
Dependent variable (output)	Velocity, v	Current, i
	Displacement, x	Charge, q
Dissipative element	Frictional coefficient of dashpot, B	Resistance, R
Storage element	Mass, M	Inductance, L
	Stiffness of spring, K	Inverse of capacitance, 1/C
Physical law	Newton's second law $\sum f = 0$	Kirchoff's voltage law $\sum v = 0$

F, M, B, k   x, v $v = \frac{dx}{dt}$

Translatory Systems Force-Current Analogy

R, L, C

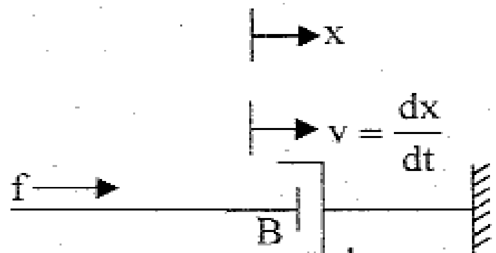
e, i $i = \frac{dq}{dt}$. $e = \frac{d\phi}{dt}$

(x,) v

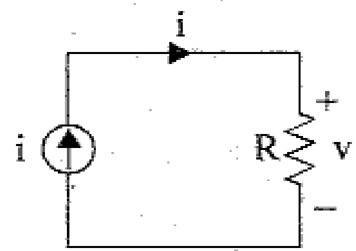
ϕ (F, 4 x)

Analogous Elements in Force-Current Analogy

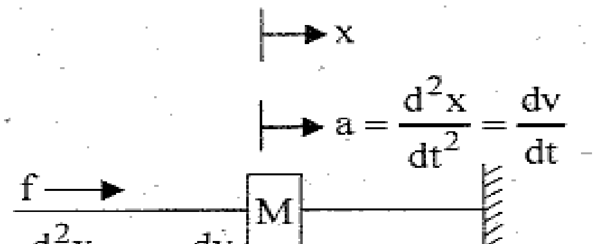
Mechanical system	Electrical system
Input : Force Output : Velocity	Input : Current source Output : Voltage across the element



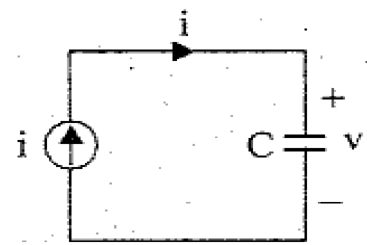
$$f = B\frac{dx}{dt} = Bv$$



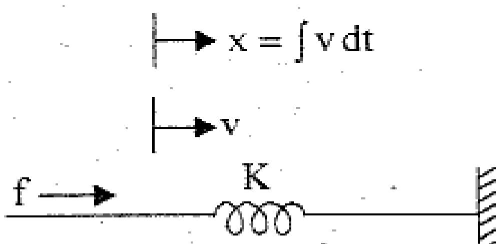
$$i = \frac{1}{R}v$$



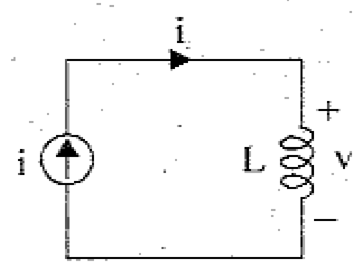
$$f = M\frac{d^2x}{dt^2} = M\frac{dv}{dt}$$



$$i = C\frac{dv}{dt}$$



$$f = Kx = K\int v\,dt$$



$$i = \frac{1}{L}\int v\,dt$$

Analogous Elements in Force-Current Analogy

Item	Mechanical system	Electrical system (node basis system)
Independent variable (input)	Force, f	Current, i
Dependent variable (output)	Velocity, v	Voltage, v
	Displacement, x	Flux, ϕ
Dissipative element	Frictional coefficient of dashpot, B	Conductance G=1/R
Storage element	Mass, M	Capacitance, C
	Stiffness of spring, K	Inverse of inductance, 1/L
Physical law	Newton's second law $\sum f = 0$	Kirchoff 's current law $\sum i = 0$

Problem 4

Force-Voltage Analogy of Following System

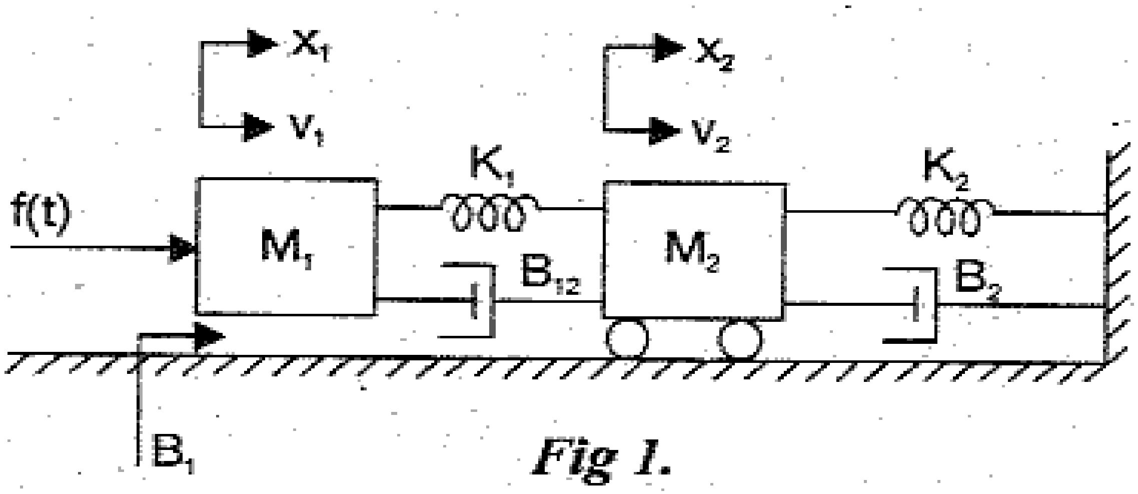


Fig 1.

$$M_1 \frac{dv_1}{dt} + B_1 v_1 + B_{12}(v_1 - v_2) + K_1 \int (v_1 - v_2)\, dt = f(t)$$

$$M_2 \frac{dv_2}{dt} + B_2 v_2 + K_2 \int v_2\, dt + B_{12}(v_2 - v_1) + K_1 \int (v_2 - v_1)\, dt = 0$$

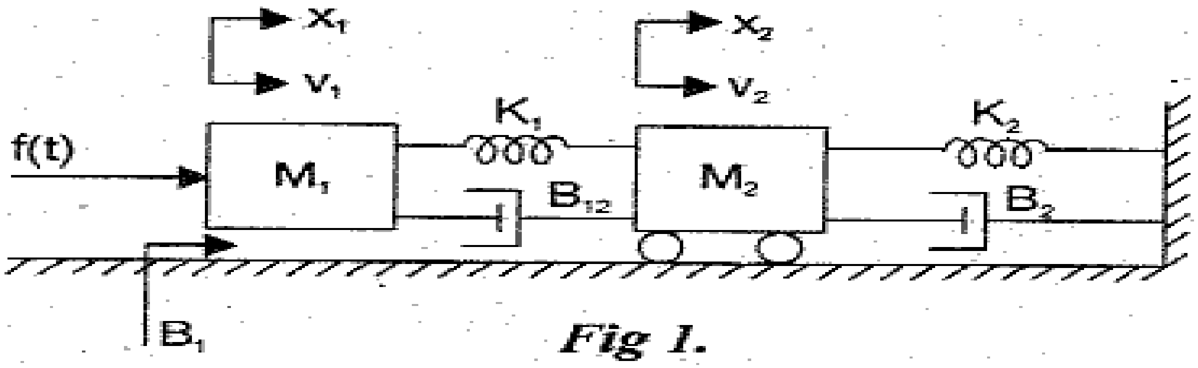
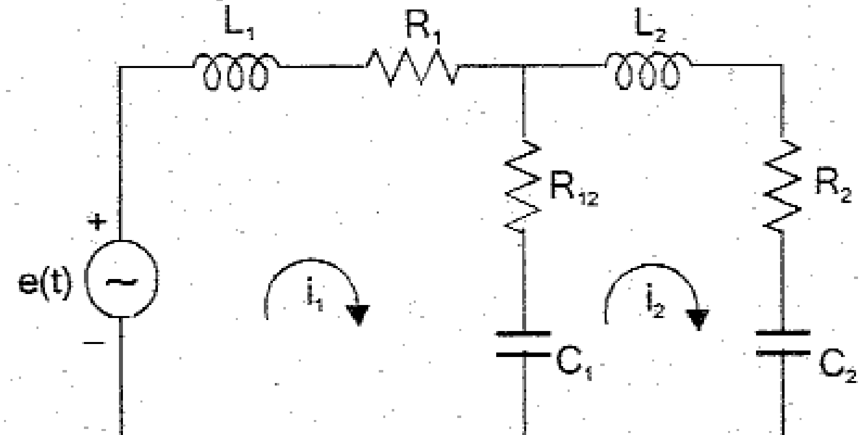


Fig 1.

$f(t) \rightarrow e(t)$	$M_1 \rightarrow L_1$	$B_1 \rightarrow R_1$	$K_1 \rightarrow 1/C_1$
$v_1 \rightarrow i_1$	$M_2 \rightarrow L_2$	$B_2 \rightarrow R_2$	$K_2 \rightarrow 1/C_2$
$v_2 \rightarrow i_2$		$B_{12} \rightarrow R_{12}$	



$$L_1\frac{di_1}{dt} + R_1 i_1 + R_{12}(i_1 - i_2) + \frac{1}{C_1}\int(i_1 - i_2)dt = e(t)$$

$$L_2\frac{di_2}{dt} + R_2 i_2 + \frac{1}{C_2}\int i_2\, dt + R_{12}(i_2 - i_1) + \frac{1}{C_1}\int(i_2 - i_1)dt = 0$$

Force-Current Analogy of Following System

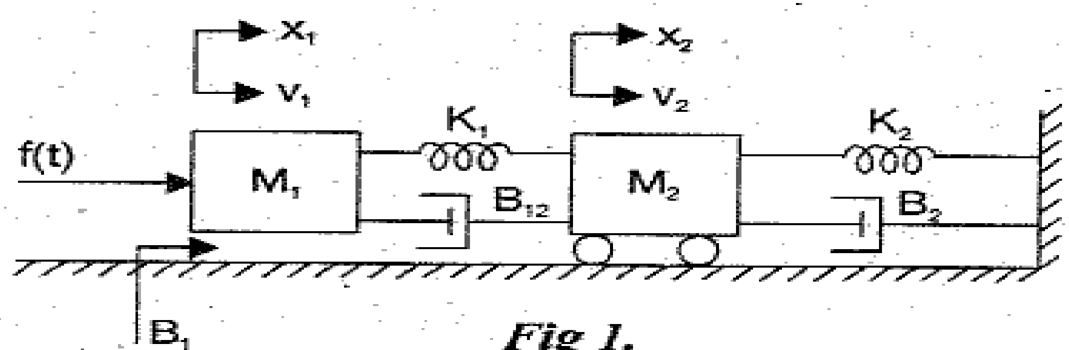


Fig 1.

$f(t) \rightarrow i(t)$

$v_1 \rightarrow v_1$

$v_2 \rightarrow v_2$

$M_1 \rightarrow C_1$

$M_2 \rightarrow C_2$

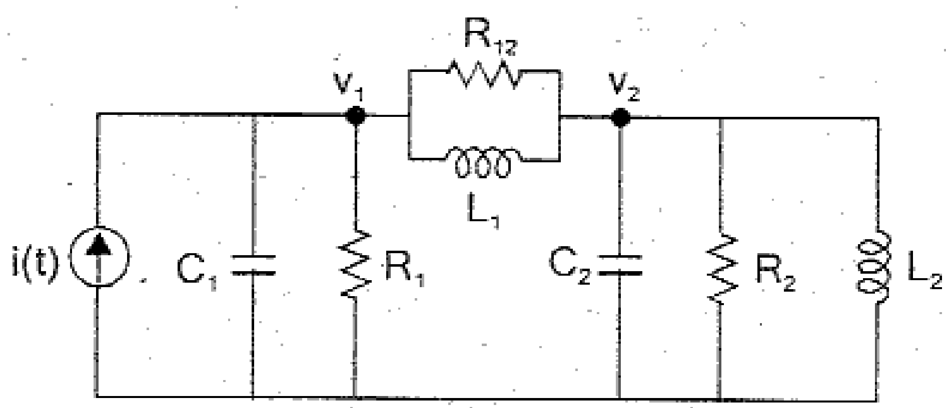
$B_{12} \rightarrow 1/R_{12}$

$B_1 \rightarrow 1/R_1$

$B_2 \rightarrow 1/R_2$

$K_1 \rightarrow 1/L_1$

$K_2 \rightarrow 1/L_2$



$$C_1\frac{dv_1}{dt}+\frac{1}{R_1}v_1+\frac{1}{R_{12}}(v_1-v_2)+\frac{1}{L_1}\int(v_1-v_2)dt=i(t)$$

$$C_2\frac{dv_2}{dt}+\frac{1}{R_2}v_2+\frac{1}{L_2}\int v_2dt+\frac{1}{R_{12}}(v_2-v_1)+\frac{1}{L_1}\int(v_2-v_1)dt=0$$

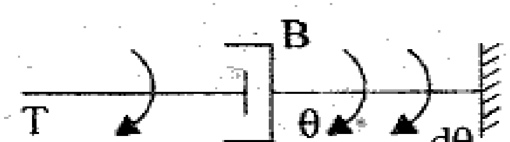
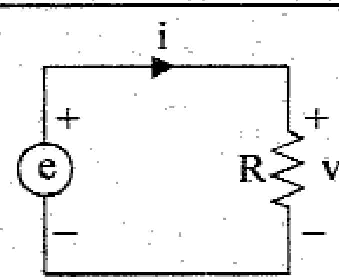
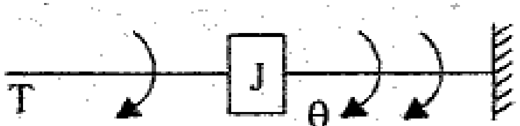
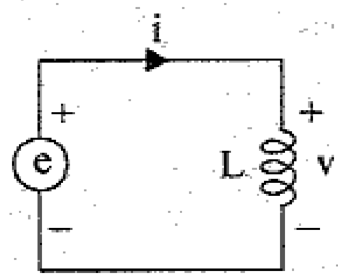
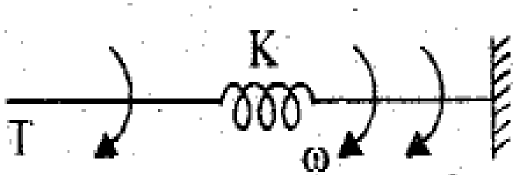
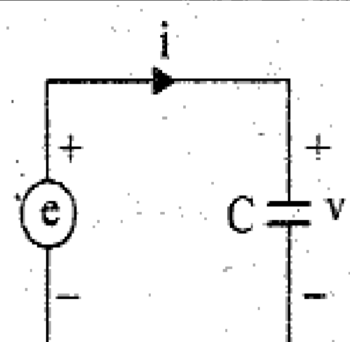
T, J, B, k θ, $\omega = \frac{d\theta}{dt}$

Rotational Systems Torque-Voltage Analogy

R, L, C

e, i $i = \frac{dq}{dt}$

Analogous Elements in Torque-VoltageAnalogy

Mechanical rotational system	**Electrical system**
Input : Torque Output : Angular velocity	Input : Voltage source Output : Current through the element
 $\omega = \frac{d\theta}{dt}$ $T = B\frac{d\theta}{dt} = B\omega$	 $e = v$; $v = Ri$ $\therefore e = Ri$
 $T = J\frac{d^2\theta}{dt^2} = J\frac{d\omega}{dt}$ $\quad \propto = \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt}$	 $e = v$; $v = L\frac{di}{dt}$ $\therefore e = L\frac{di}{dt}$
 $\theta = \int \omega\, dt$ $T = K\theta = K\int \omega\, dt$	 $e = v$; $v = \frac{1}{C}\int i\,dt$ $\therefore e = \frac{1}{C}\int i\,dt$

Analogous Elements in Torque-VoltageAnalogy

Item	Mechanical rotational system	Electrical system (mesh basis system)
Independent variable (input)	Torque, T	Voltage, e, v
Dependent variable (output)	Angular Velocity, ω	Current, i
	Angular displacement, θ	Charge, q
Dissipative element	Rotational coefficient of dashpot, B	Resistance, R
Storage element	Moment of inertia, J	Inductance, L
	Stiffness of spring, K	Inverse of capacitance, 1/C
Physical law	Newton's second law $\sum T = 0$	Kirchoff's voltage law $\sum v = 0$

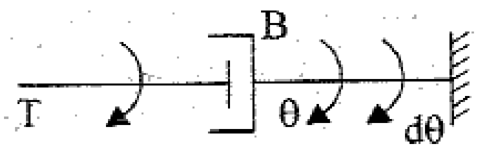
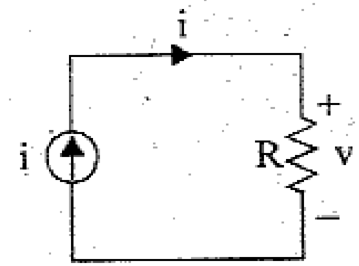
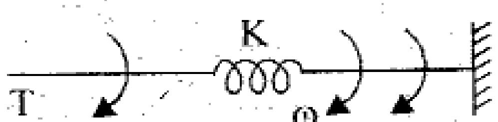
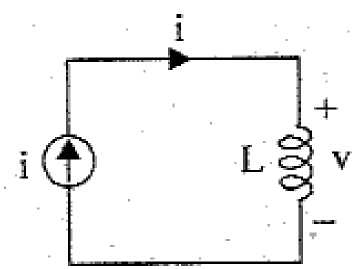
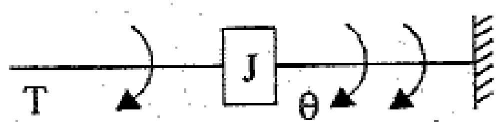
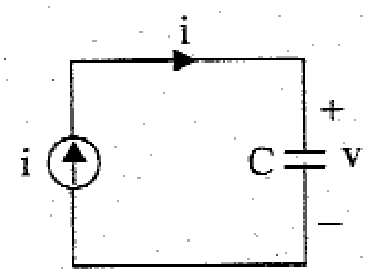
T, J, B, k $\theta, \omega = \frac{d\theta}{dt}$

Rotational Systems Torque-Current Analogy

$R \quad L \quad C$

$e, i \quad i = \frac{dq}{dt} \quad e = \frac{d\phi}{dt}$

Analogous Elements in Torque-Current Analogy

Mechanical rotational system	Electrical system
Input : Torque Output : Angular velocity	Input : Current source Output : Voltage across the element
 $\omega = \frac{d\theta}{dt}$ $T = B\frac{d\theta}{dt} = B\omega$	 $i = \frac{1}{R}v$
 $\theta = \int \omega\, dt$ $T = K\theta = K\int \omega\, dt$	 $i = \frac{1}{L}\int v\, dt$
 $T = J\frac{d^2\theta}{dt^2} = J\frac{d\omega}{dt}$ $\propto = \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt}$	 $i = C\frac{dv}{dt}$

Analogous Elements in Torque-Current Analogy

Item	Mechanical rotational system	Electrical system (node basis system)
Independent variable (input)	Torque, T	Current, i
Dependent variable (output)	Angular Velocity, ω	Voltage, v
	Angular displacement, θ	Flux, ϕ
Dissipative element	Rotational frictional coefficient of dashpot, B	Conductance, $G = 1/R$
Storage element	Moment of inertia, J	Capacitance, C
	Stiffness of spring, K	Inverse of inductance, 1/L
Physical law	Newton's second law $\sum T = 0$	Kirchoff 's current law $\sum i = 0$

Torque-Voltage Analogy of Following System

Problem 5

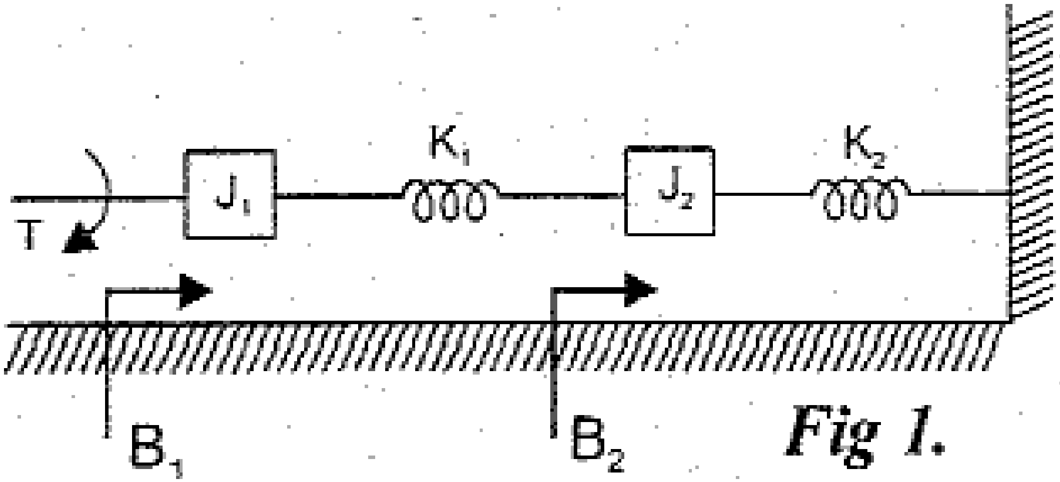


Fig 1.

$$J_1\frac{d\omega_1}{dt}+B_1\omega_1+K_1\int(\omega_1-\omega_2)dt=T$$

$$J_2\frac{d\omega_2}{dt}+B_2\omega_2+K_2\int\omega_2 dt+K_1\int(\omega_2-\omega_1)dt=0$$

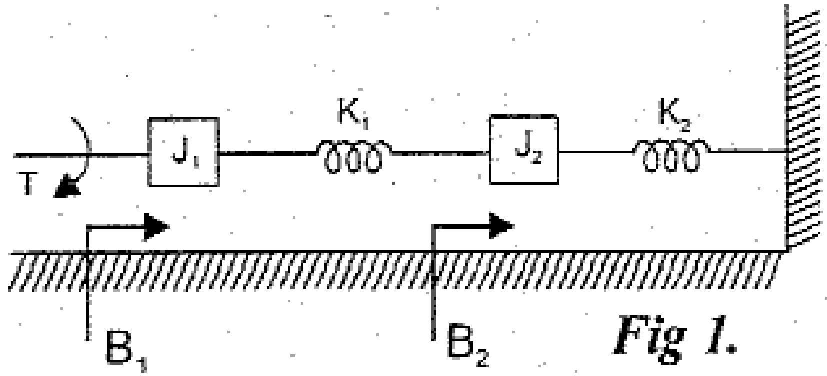
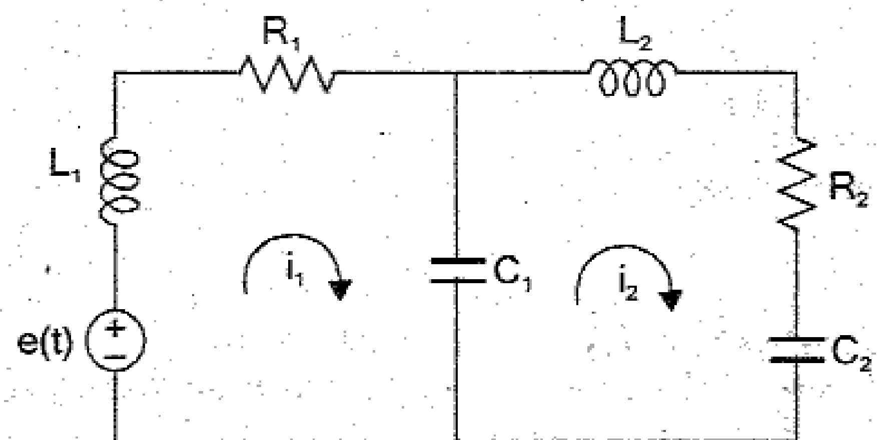


Fig 1.

$T \rightarrow e(t)$	$J_1 \rightarrow L_1$	$B_1 \rightarrow R_1$	$K_1 \rightarrow 1/C_1$
$\omega_1 \rightarrow i_1$	$J_2 \rightarrow L_2$	$B_2 \rightarrow R_2$	$K_2 \rightarrow 1/C_2$
$\omega_2 \rightarrow i_2$			



$$L_1\frac{di_1}{dt}+R_1i_1+\frac{1}{C_1}\int(i_1-i_2)=e(t)$$

$$L_2\frac{di_2}{dt}+R_2i_2+\frac{1}{C_2}\int i_2dt+\frac{1}{C_1}\int(i_2-i_1)dt=0$$

Torque-Current Analogy of Following System

$T \to i(t)$ $B_1 \to 1/R_1$ $\omega_1 \to v_1$ $J_1 \to C_1$ $K_1 \to 1/L_1$

$B_2 \to 1/R_2$ $\omega_2 \to v_2$ $J_2 \to C_2$ $K_2 \to 1/L_2$

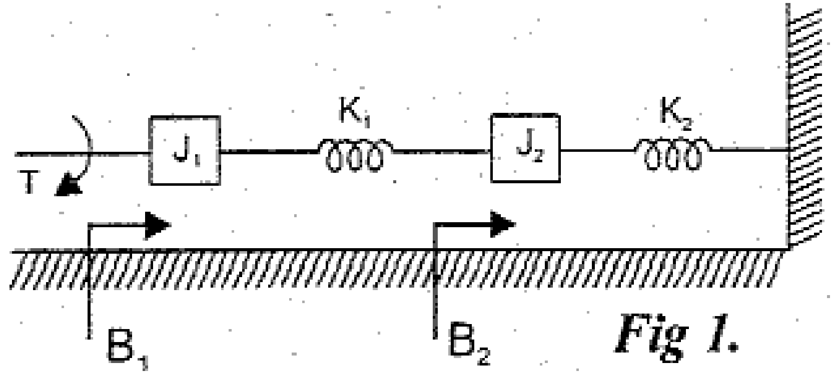
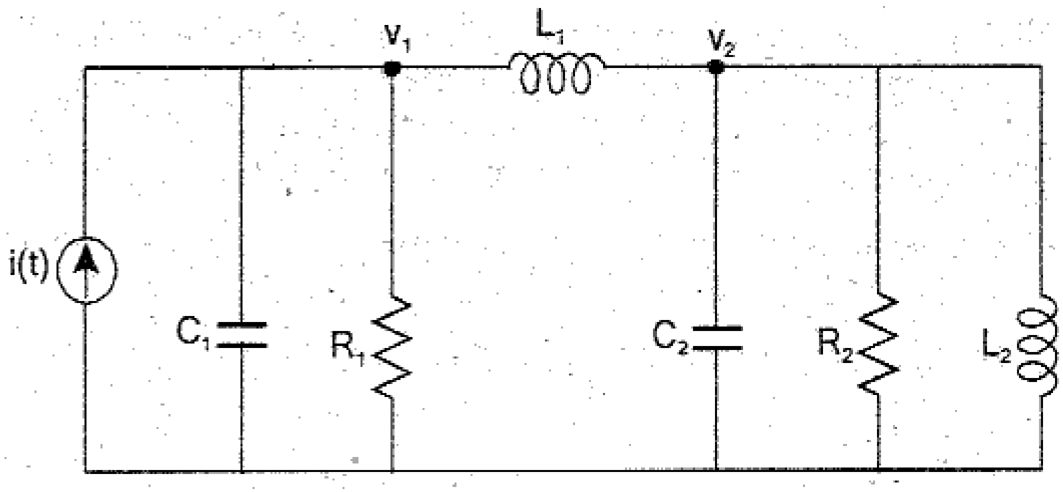


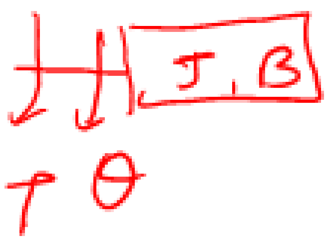
Fig 1.



$$C_1\frac{dv_1}{dt} + \frac{1}{R_1}v_1 + \frac{1}{L_1}\int(v_1 - v_2)dt = i(t)$$

$$C_2\frac{dv_2}{dt} + \frac{1}{R_2}v_2 + \frac{1}{L_2}\int v_2 dt + \frac{1}{L_1}\int(v_2 - v_1)dt = 0$$

Armature and Field Controlled DC Motor



Motor

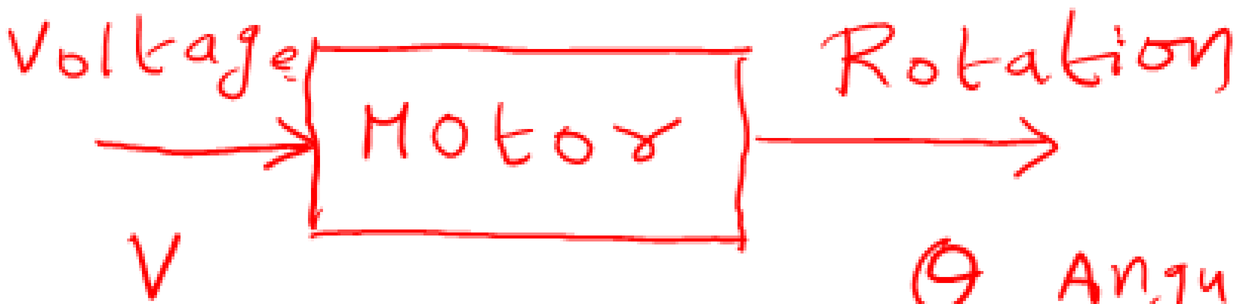
Input is Voltage

Output is Rotation

$$J\frac{d^2\theta}{dt^2}$$

$$B\frac{d\theta}{dt}$$

$$K\theta$$

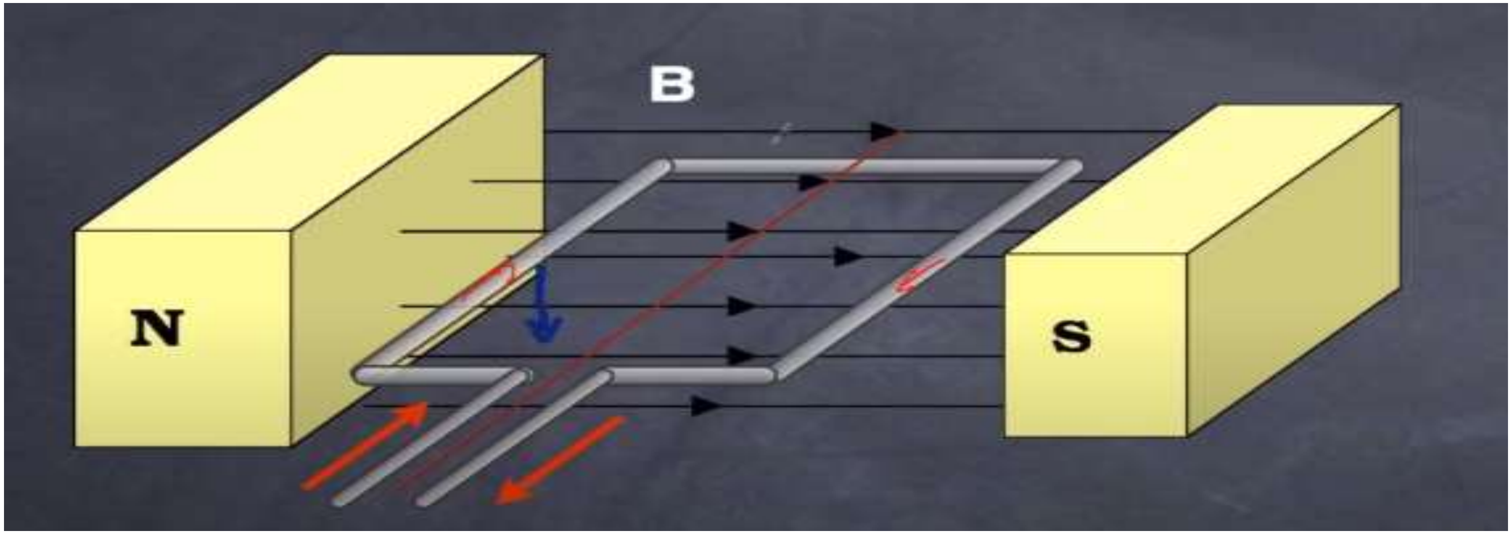


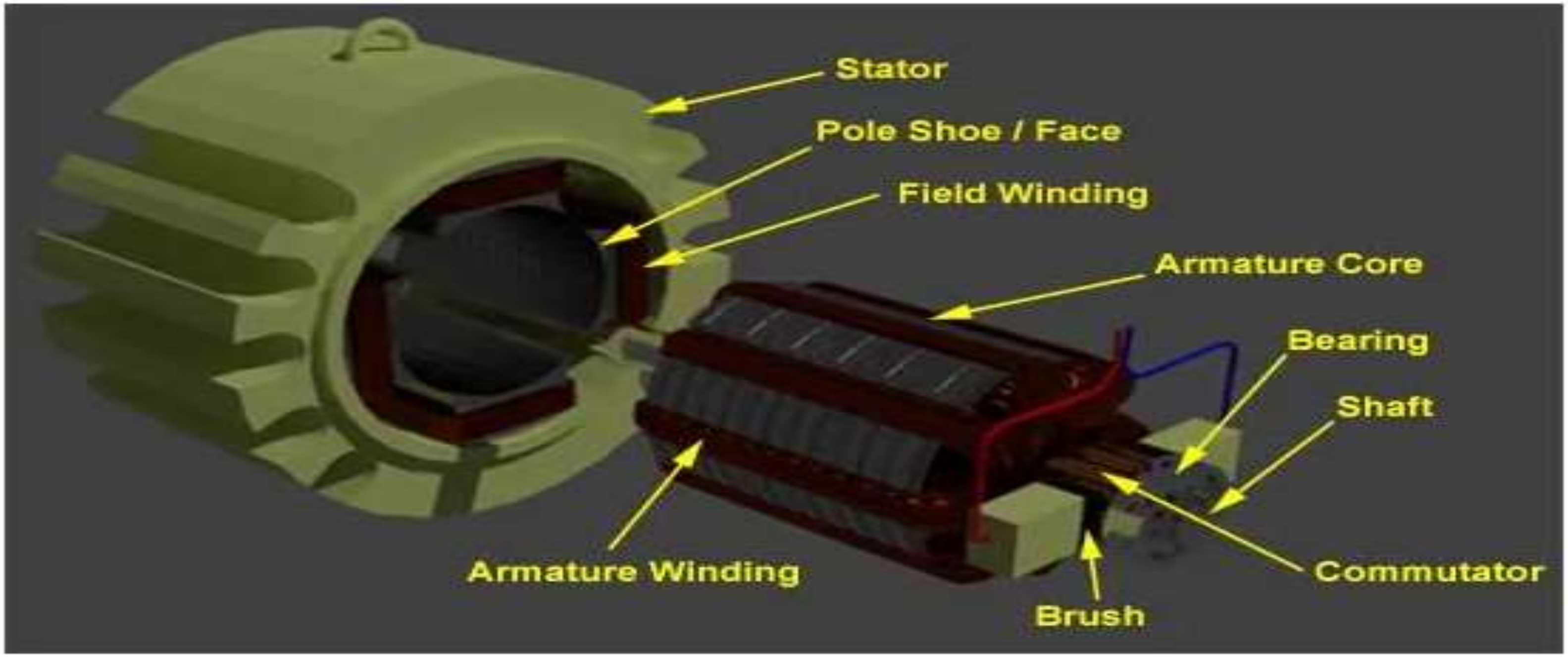
θ Angular Displacement

(or)

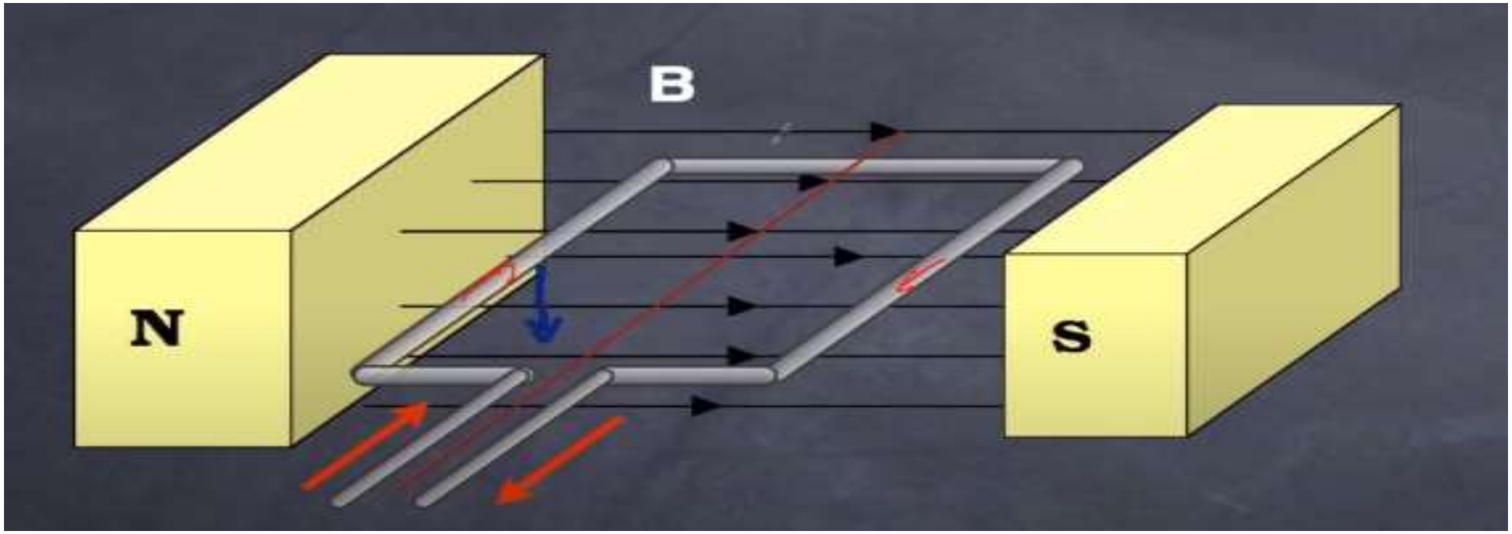
ω Angular velocity

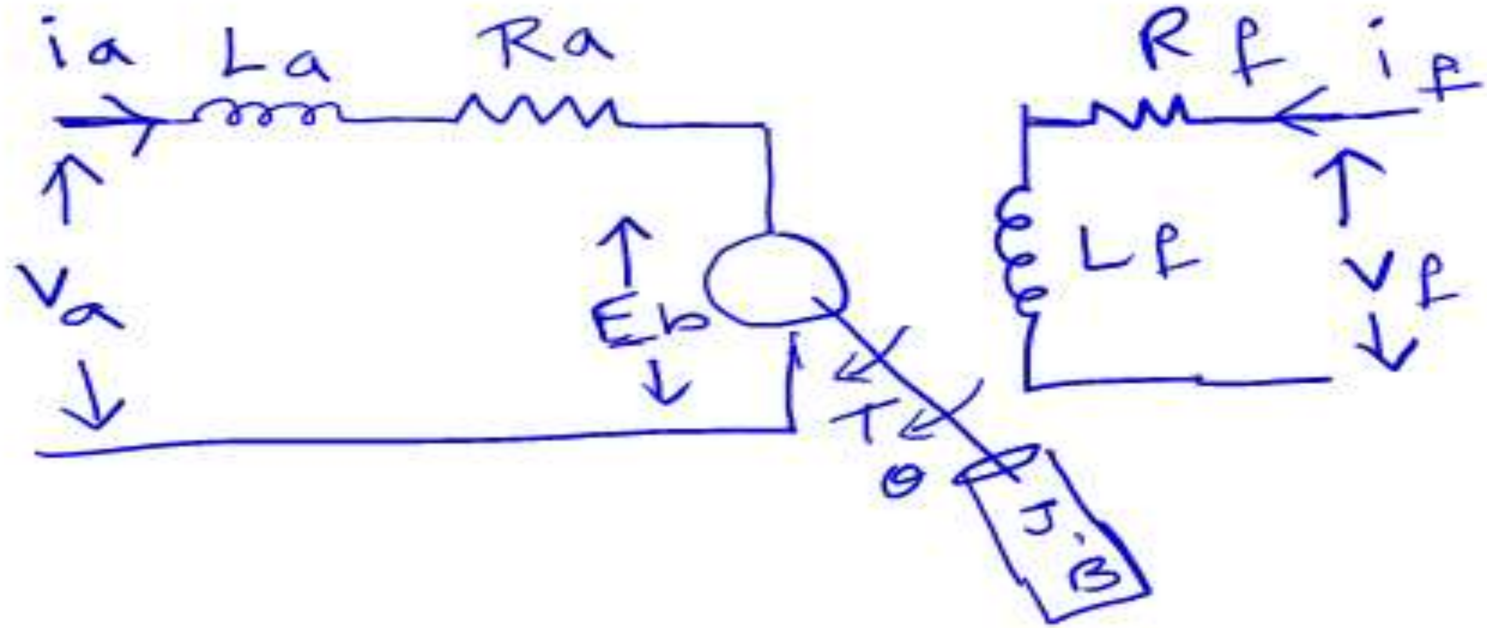
DC MOTOR

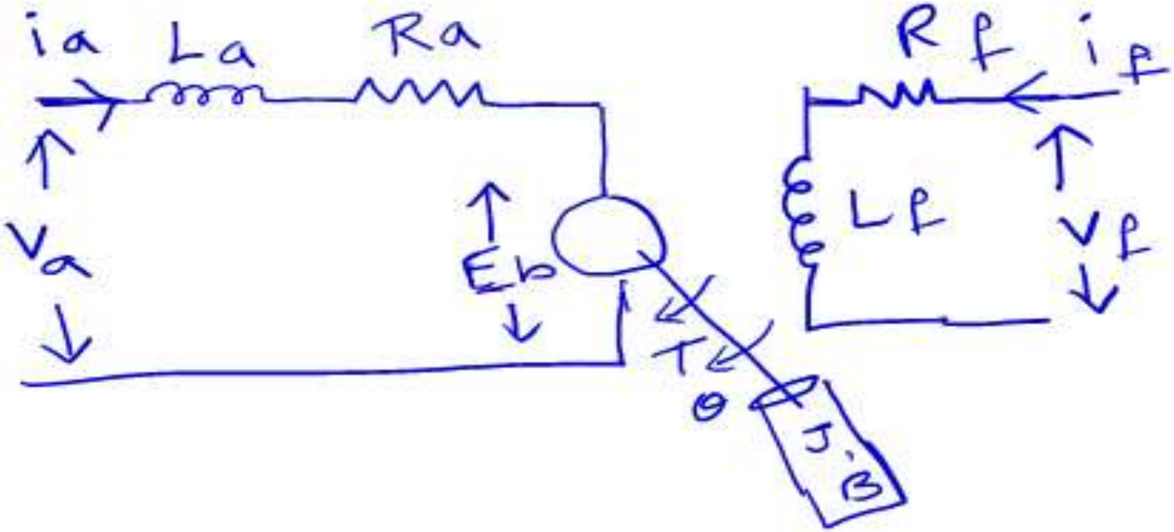

B
N
S


Stator
Pole Shoe / Face
Field Winding
Armature Core
Bearing
Shaft
Commutator
Brush
Armature Winding

DC MOTOR







Let, R_a = Arm

L_a = Armature inductance, H

i_a = Armature current, A

v_a = Armature voltage, V

e_b = Back emf, V

K_t = Torque constant, N-m/A

T = Torque developed by motor, N-m

θ = Angular displacement of shaft, rad

J = Moment of inertia of motor and load, Kg-m²/rad

B = Frictional coefficient of motor and load, N-m/(rad/sec)

K_b = Back emf constant, V/(rad/sec)

R_f = Field resistance, Ω

L_f = Field inductance, H

i_f = Field current, A

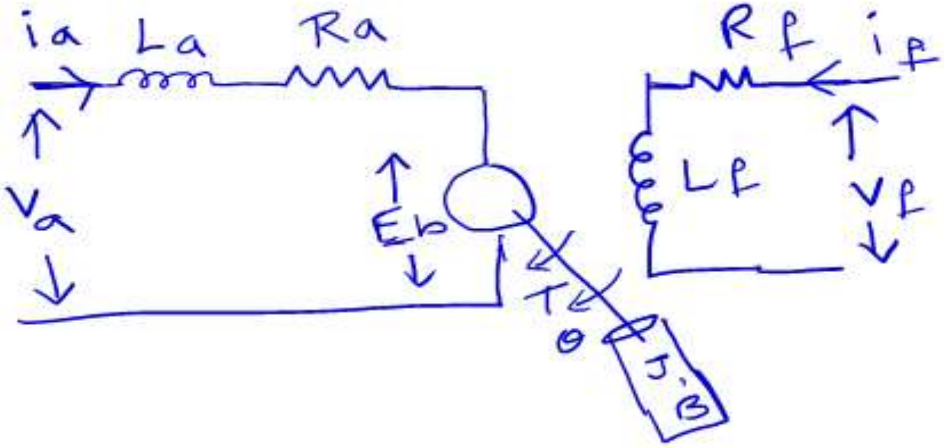
v_f = Field voltage, V

T = Torque developed by motor, N-m

K_{tf} = Torque constant, N-m/A

Armature Controlled DC Motor

$$i_a R_a + L_a \frac{di_a}{dt} + e_b = v_a$$



$$T \propto i_a$$

. Torque, $T = K_t\, i_a$

$$J \frac{d^2\theta}{dt^2} + B \frac{d\theta}{dt} = T$$

$$I_a(s)\, R_a + L_a s\, I_a(s) + E_b(s) = V_a(s)$$

$$T(s) = K_t I_a(s)$$

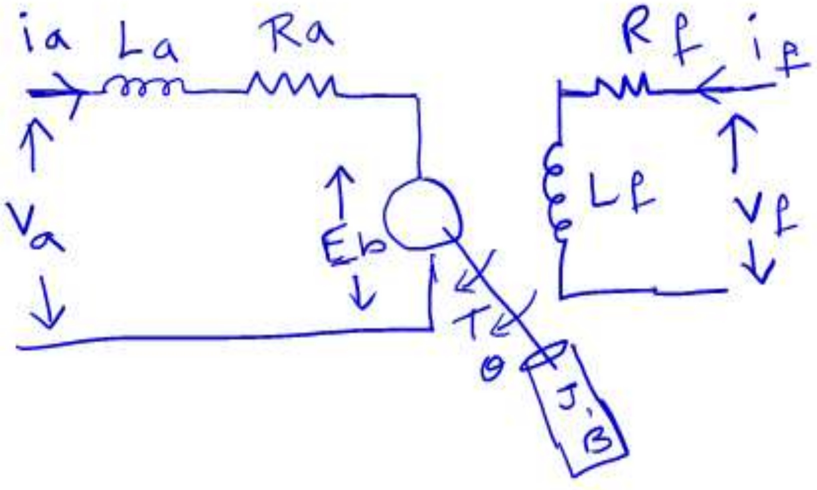
$$e_b \propto \frac{d\theta}{dt}$$

$$Js^2\theta(s) + B\, s\, \theta(s) = T(s)$$

$$E_b(s) = K_b s\, \theta(s)$$

Back emf, $e_b = K_b \frac{d\theta}{dt}$

Field Controlled DC Motor



$$R_f i_f + L_f \frac{di_f}{dt} = v_f$$

$$T \propto i_f$$

$$\text{Torque, } T = K_{tf}\, i_f$$

$$J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = T$$

$$R_f I_f(s) + L_f s\, I_f(s) = V_f(s)$$

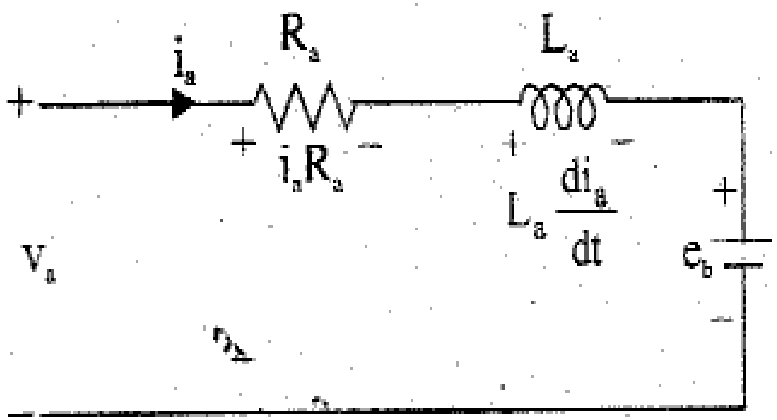
$$T(s) = K_{tf}\, I_f(s)$$

$$Js^2\theta(s) + Bs\theta(s) = T(s)$$

Armature Controlled DC Motor

Transfer Function for Armature Controlled DC Motor

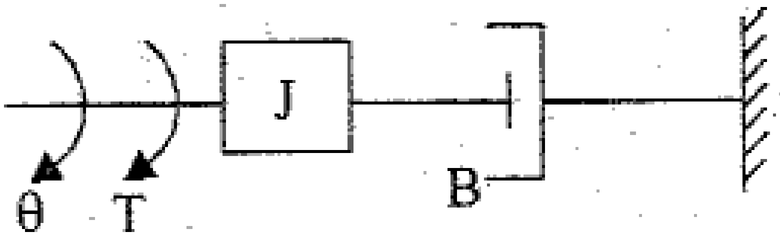
The equivalent circuit of armature is shown



By Kirchoff's voltage law, we can write,

$$i_a R_a + L_a \frac{di_a}{dt} + e_b = v_a$$

Torque of DC motor is proportional to the product of flux and current. Since flux is constant in this system, the torque is proportional to i_a alone.



$$T \propto i_a$$

$$\therefore \text{ Torque, } T = K_t\, i_a$$

The differential equation governing the mechanical system of motor is given by,

$$J \frac{d^2\theta}{dt^2} + B \frac{d\theta}{dt} = T$$

The back emf of DC machine is proportional to speed (angular velocity) of shaft.

$$\therefore e_b \propto \frac{d\theta}{dt} \quad \text{or} \quad \text{Back emf, } e_b = K_b \frac{d\theta}{dt}$$

The Laplace transform of various time domain signals involved in this system are shown below.

$$\mathcal{L}\{v_a\} = V_a(s); \quad \mathcal{L}\{e_b\} = E_b(s); \quad \mathcal{L}\{T\} = T(s); \quad \mathcal{L}\{i_a\} = I_a(s); \quad \mathcal{L}\{\theta\} = \theta(s)$$

The differential equations governing the armature controlled DC motor speed control system are,

$$i_a R_a + L_a \frac{di_a}{dt} + e_b = v_a \quad ; \quad T = K_t i_a \quad ; \quad J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = T \quad ; \quad e_b = K_b \frac{d\theta}{dt}$$

$$i_a R_a + L_a \frac{di_a}{dt} + e_b = v_a \quad ; \quad T = K_t i_a \quad ; \quad J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = T \quad ; \quad e_b = K_b \frac{d\theta}{dt}$$

Taking Laplace transform of the above equations

$$I_a(s)\, R_a + L_a s I_a(s) + E_b(s) = V_a(s) \quad \text{— (1)} \qquad (R_a + sL_a)\, I_a(s) + E_b(s) = V_a(s) \quad \text{— (2)}$$

$$T(s) = K_t I_a(s) \quad \text{— (3)}$$

$$Js^2\theta(s) + B\, s\, \theta(s) = T(s) \quad \text{— (4)}$$

$$E_b(s) = K_b s\, \theta(s) \quad \text{— (5)}$$

From (3) & (4)

$$K_t I_a(s) = (Js^2 + Bs)\, \theta(s)$$

$$I_a(s) = \frac{(Js^2 + Bs)}{K_t}\, \theta(s) \quad \text{— (6)}$$

Substitute (5) & (6) in (2)

$$(R_a + sL_a)\ \frac{(Js^2 + Bs)}{K_t}\theta(s) + K_b s\ \theta(s) = V_a(s)$$

$$\left[\frac{(R_a + sL_a)\ (Js^2 + Bs) + K_b K_t s}{K_t}\right]\theta(s) = V_a(s)$$

The required transfer function is $\frac{\theta(s)}{V_a(s)}$

$$\therefore \frac{\theta(s)}{V_a(s)} = \frac{K_t}{(R_a + sL_a)\ (Js^2 + Bs) + K_b K_t s}$$

Block Diagram for Armature Controlled DC Motor

$$I_a(s)\,R_a + L_a s\,I_a(s) + E_b(s) = V_a(s) \quad \text{--- (1)}$$

$$T(s) = K_t I_a(s) \quad \text{--- (2)}$$

$$Js^2\theta(s) + B\,s\,\theta(s) = T(s) \quad \text{--- (3)}$$

$$E_b(s) = K_b s\,\theta(s) \quad \text{--- (4)}$$

(1) $V_a(s) - E_b(s) = I_a(s)\left[R_a + L_a s\right]$

$$\frac{1}{R_a + L_a s} = \frac{I_a(s)}{V_a(s) - E_b(s)}$$

$$T(s) = \frac{O/P}{I/P}$$

$$I_a(s)\,R_a + L_a s\,I_a(s) + E_b(s) = V_a(s) \quad \text{— (1)}$$

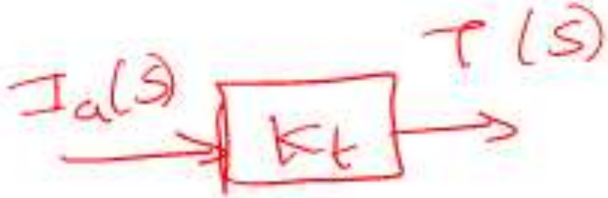
$$T(s) = K_t I_a(s) \quad \text{— (2)}$$

$$Js^2\theta(s) + B\,s\,\theta(s) = T(s) \quad \text{— (3)}$$

$$E_b(s) = K_b s\,\theta(s) \quad \text{— (4)}$$

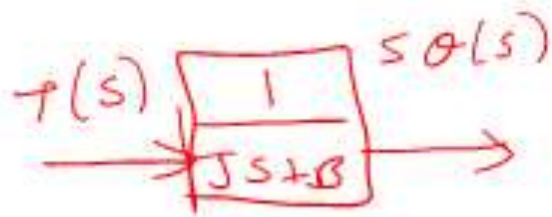
$$\frac{O/P}{I/P} = T.F$$

(2) $\dfrac{T(s)}{I_a(s)} = K_t$



(3) $(Js+B)\,s\theta(s) = T(s)$

$$\frac{s\theta(s)}{T(s)} = \frac{1}{Js+B}$$



Block Diagram for Armature Controlled DC Motor

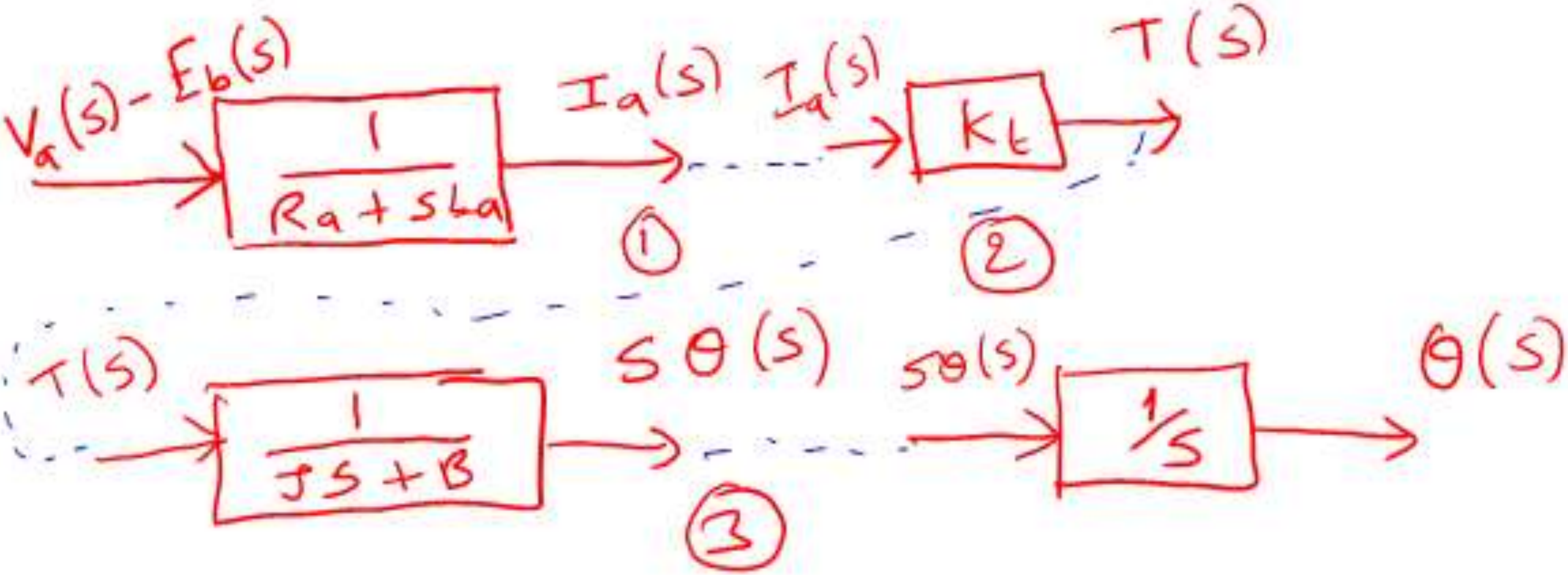
$$I_a(s)\,R_a + L_a s I_a(s) + E_b(s) = V_a(s) \quad -\ (1)$$

$$T(s) = K_t I_a(s) \quad -\ (2)$$

$$Js^2\theta(s) + B\,s\,\theta(s) = T(s) \quad -\ (3)$$

$$E_b(s) = K_b s\,\theta(s) \quad -\ (4)$$

$$\frac{O/P}{I/P} = T.F$$



Block Diagram for Armature Controlled DC Motor

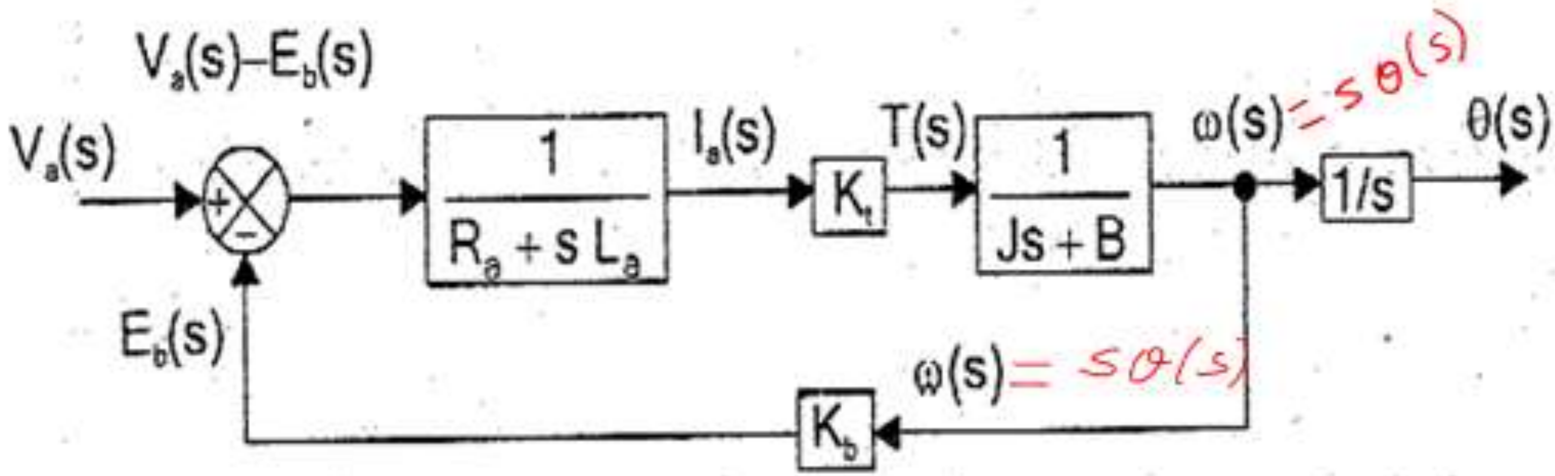
$$I_a(s)\,R_a + L_a s I_a(s) + E_b(s) = V_a(s)$$

$$T(s) = K_t I_a(s)$$

$$Js^2\theta(s) + B\,s\,\theta(s) = T(s) \Rightarrow (Js+B)\,s\theta(s) = T(s)$$

I/P O/P

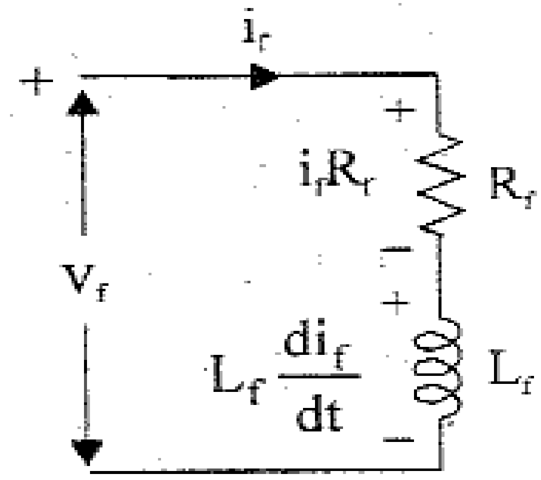
$$E_b(s) = K_b s\,\theta(s)$$



Field Controlled DC Motor

Transfer Function for Field Controlled DC Motor

The equivalent circuit of field is shown in fig

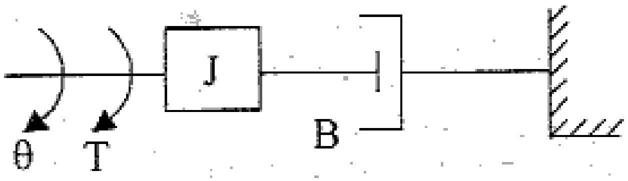


By Kirchoff 's voltage law, we can write

$$R_f i_f + L_f \frac{di_f}{dt} = v_f$$

$$T \propto i_f, \quad \therefore \text{Torque, } T = K_{tf}\, i_f$$

The differential equation governing



$$J \frac{d^2\theta}{dt^2} + B \frac{d\theta}{dt} = T$$

The Laplace transform of various time domain signals involved in this system are sho

$\mathcal{L}\{i_f\} = I_f(s)$; $\mathcal{L}\{T\} = T(s)$; $\mathcal{L}\{v_f\} = V_f(s)$; $\mathcal{L}\{\theta\} = \theta(s)$

The differential equations governing the field controlled DC motor are,

$$K_f i_f + L_f \frac{di_f}{dt} = v_f \quad ; \quad T = K_{tf} i_f \quad ; \quad J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = T$$

On taking Laplace transform of the above equations with zero initial condition we get,

$$R_f I_f(s) + L_f s\, I_f(s) = V_f(s) \quad \text{— (1)}$$

$$(R_f + sL_f)\, I_f(s) = V_f(s) \quad \text{— (2)}$$

$$T(s) = K_{tf}\, I_f(s) \quad \text{— (3)}$$

$$Js^2\theta(s) + Bs\theta(s) = T(s) \quad \text{— (4)}$$

(3) & (4)

$$K_{tf}\, I_f(s) = Js^2\theta(s) + Bs\theta(s)$$

$$I_f(s) = s\,\frac{(Js + B)}{K_{tf}}\theta(s) \quad \text{— (5)}$$

substitute (5) in (2)

$$(R_f + sL_f)s\,\frac{(Js + B)}{K_{tf}}\,\theta(s) = V_f(s)$$

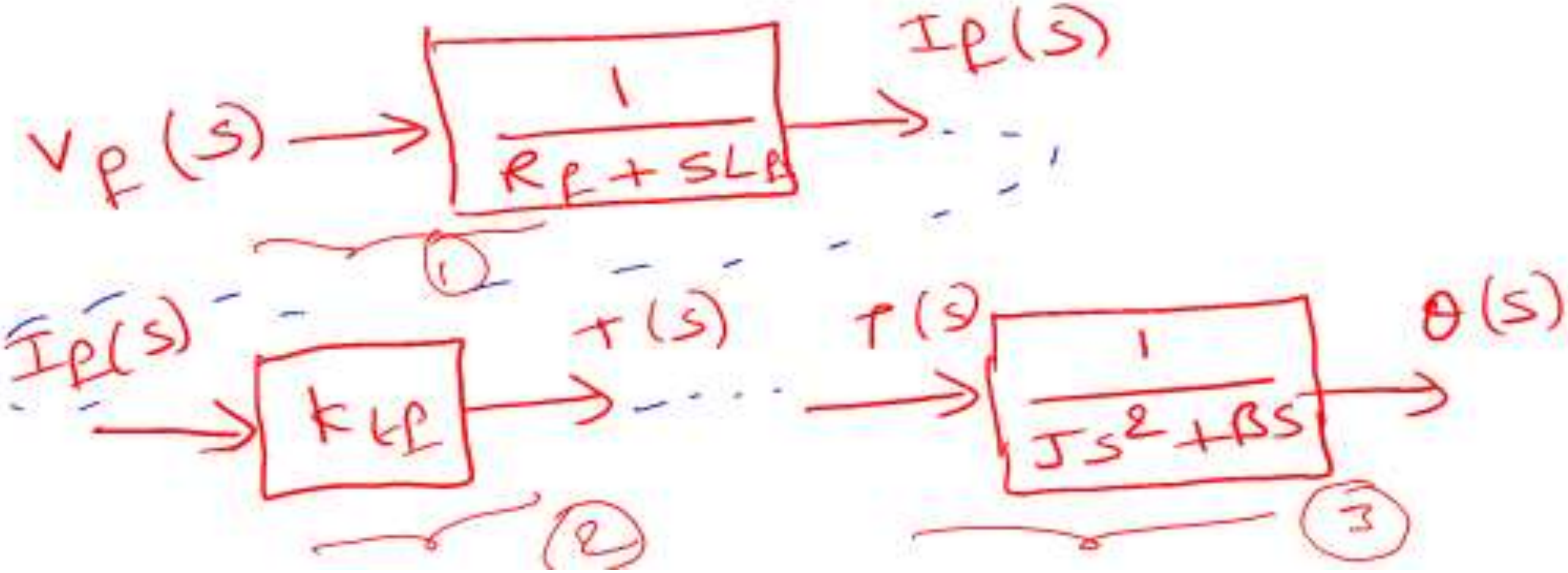
$$\frac{\theta(s)}{V_f(s)} = \frac{K_{tf}}{s\,(R_f + sL_f)\,(B + sJ)}$$

Block Diagram for Field Controlled DC Motor

$$R_f I_f(s) + L_f s I_f(s) = V_f(s) \quad \text{— (1)}$$

$$T(s) = K_{tf} I_f(s) \quad \text{— (2)}$$

$$Js^2\theta(s) + Bs\theta(s) = T(s) \quad \text{— (3)}$$

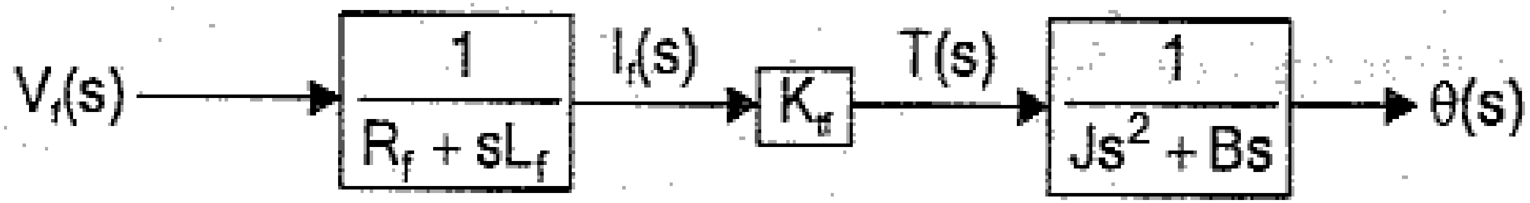


Block Diagram for Field Controlled DC Motor

$$R_f I_f(s) + L_f s I_f(s) = V_f(s)$$

$$T(s) = K_{tf} I_f(s)$$

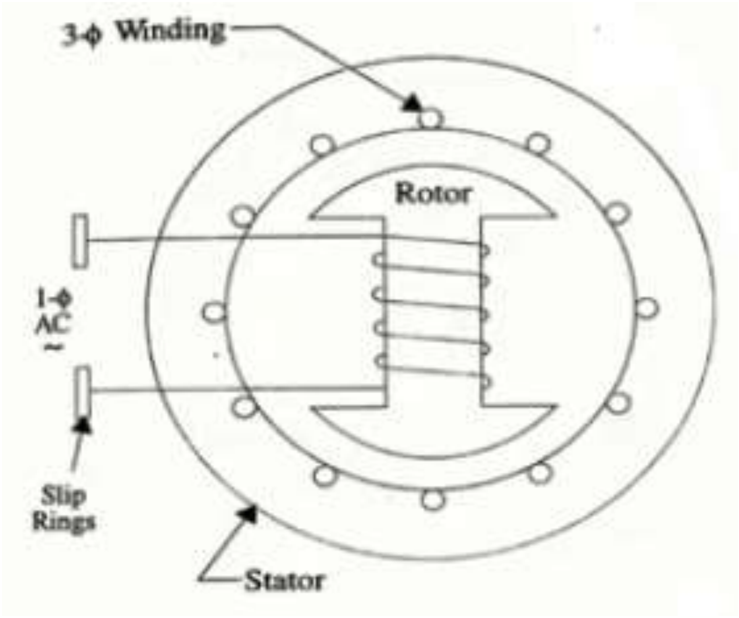
$$Js^2\theta(s) + Bs\theta(s) = T(s)$$

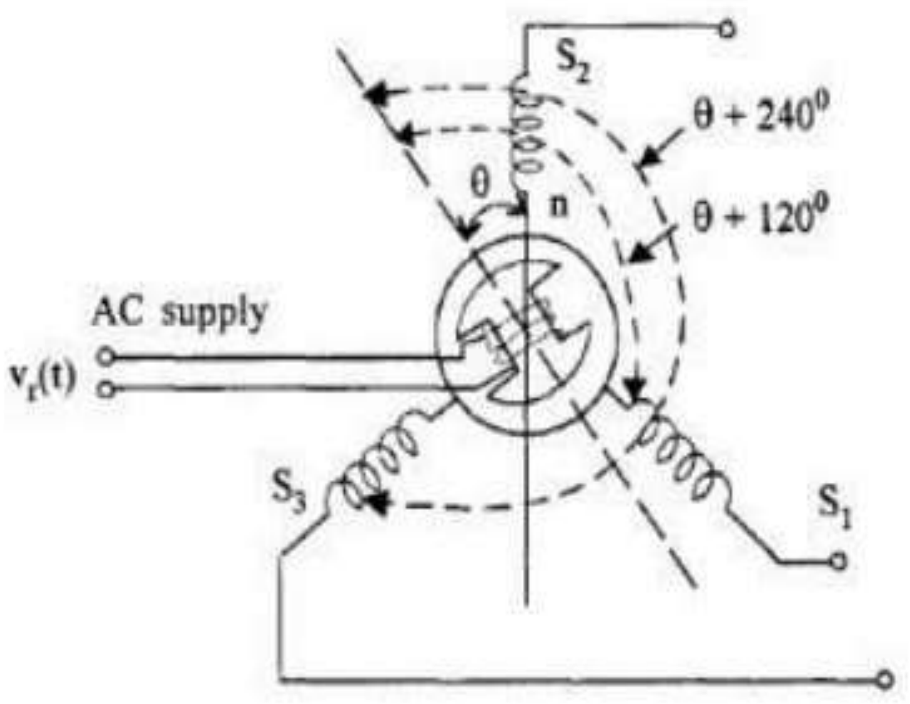


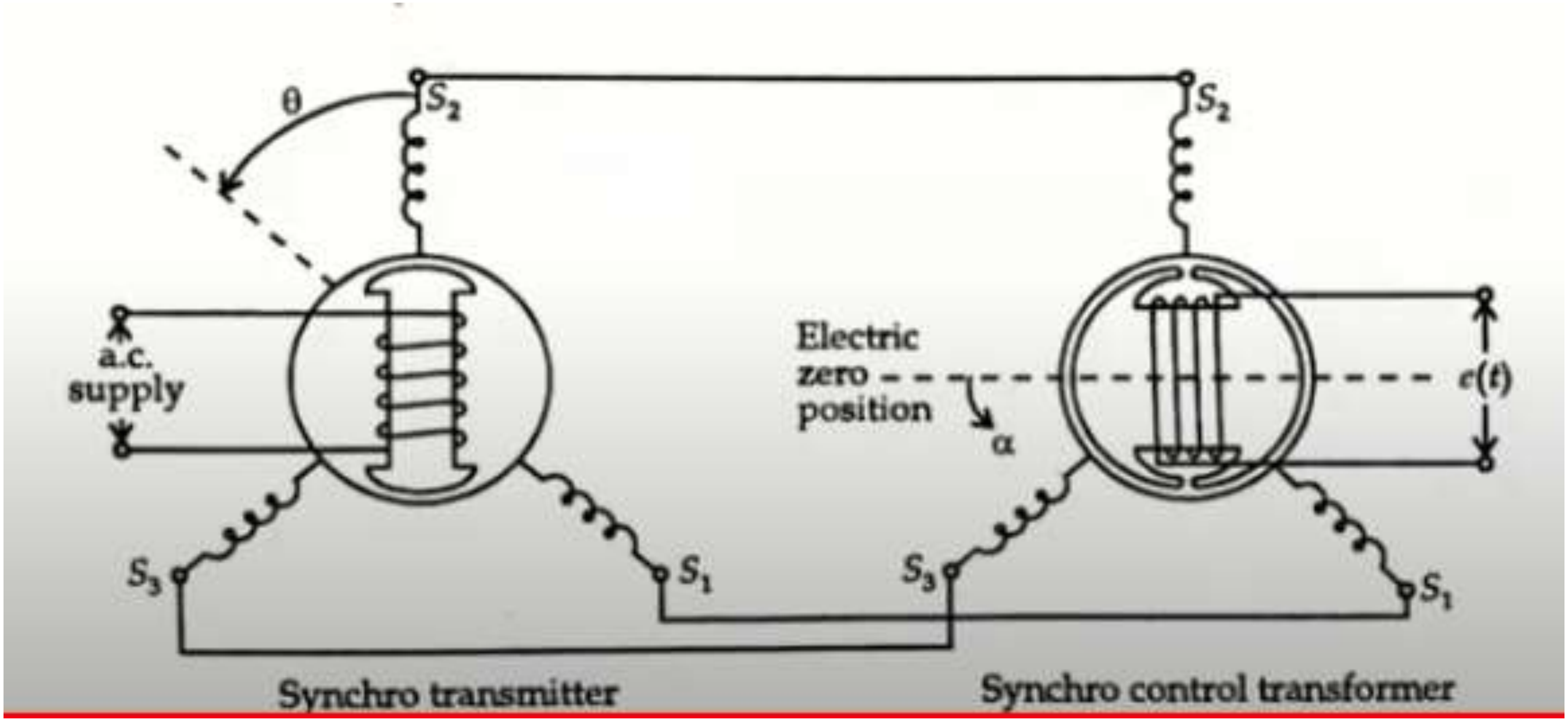
Synchro Transmitter and Receiver

Synchro Transmitter and Receiver

The **Synchro** is a type of transducer which transforms the angular position of the shaft into an electric signal. It is used as an error detector and as a rotary position sensor







Here e(t) voltage is obtained based on angular position of Rotor

TIME RESPONSE ANALYSIS

TEST SIGNALS

The characteristics of input signals are

1) Sudden Shock
2) A Sudden change
3) A constant Velocity
4) Constant Acceleration

To study the system behaviour in labolatory we use test signals which these characteristics and are used as input signals to predict the performance of system.

The Commonly used Test Signals are

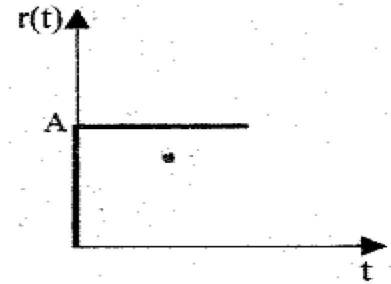
1) Step Signal (Steady Input)
2) Ramp Signal (Increases linearly with time- Constant Acceleration)
3) Parabolic Signal (Constant acceleration)
4) Impulse Signal. (Sudden Shock)

TEST SIGNALS

STEP SIGNAL

The mathematical representation

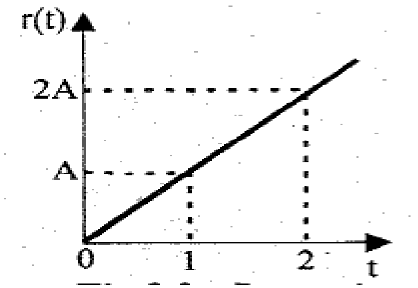
$$r(t) = 1 \; ; \; t \geq 0$$
$$= 0 \; ; \; t < 0$$



RAMP SIGNAL

The mathematical representation

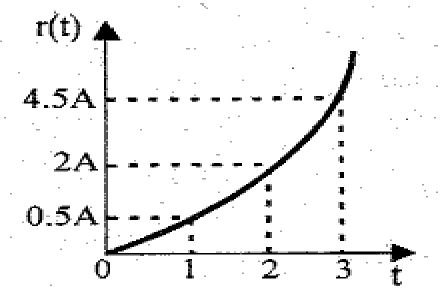
$$r(t) = A\,t \; ; \; t \geq 0$$
$$= 0 \; ; \; t < 0$$



PARABOLIC SIGNAL

The mathematical representation

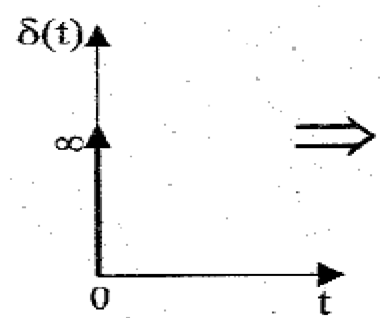
$$r(t) = \frac{At^2}{2} \; ; \; t \geq 0$$
$$= 0 \; ; \; t < 0$$



IMPULSE SIGNAL

$$\delta(t) = \infty; \; t = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t)dt = A$$

$$= 0; \; t \neq 0$$



S-Domain Representation of TEST Signals

Name of the signal	Time domain equation of signal, r(t)	Laplace transform of the signal, R(s)
Step	A	$\frac{A}{s}$
Unit step	1	$\frac{1}{s}$
Ramp	At	$\frac{A}{s^2}$
Unit ramp	t	$\frac{1}{s^2}$
Parabolic	$\frac{At^2}{2}$	$\frac{A}{s^3}$
Unit parabolic	$\frac{t^2}{2}$	$\frac{1}{s^3}$
Impulse	$\delta(t)$	1

ORDER OF A SYSTEM

$$\text{Transfer function, } T(s) = \frac{P(s)}{Q(s)} = \frac{b_0 s^m + b_1 s^{m-1} + b_2 s^{m-2} + \dots\dots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots\dots + a_{n-1} s + a_n}$$

When $n = 0$, the system is zero order system.

When $n = 1$, the system is first order system.

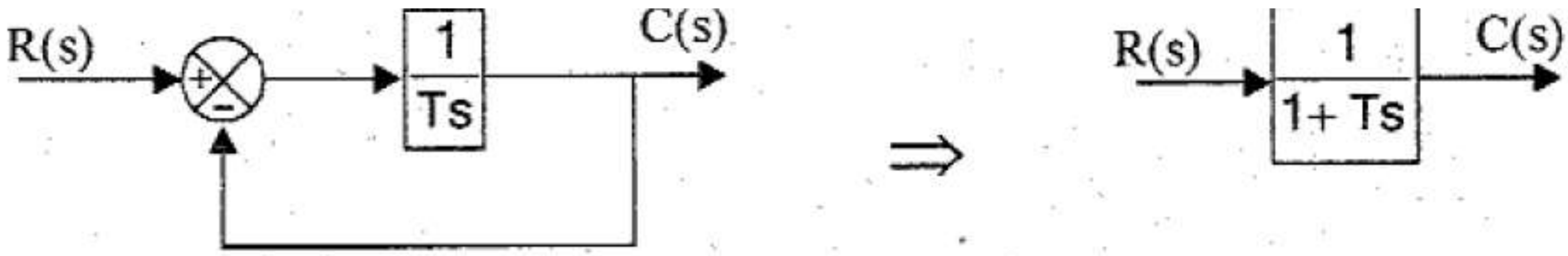
When $n = 2$, the system is second order system and so on

$$T(s) = \frac{P(s)}{Q(s)} = \frac{(s+z_1)(s+z_2)\dots\dots(s+z_m)}{(s+p_1)(s+p_2)\dots\dots(s+p_n)}$$

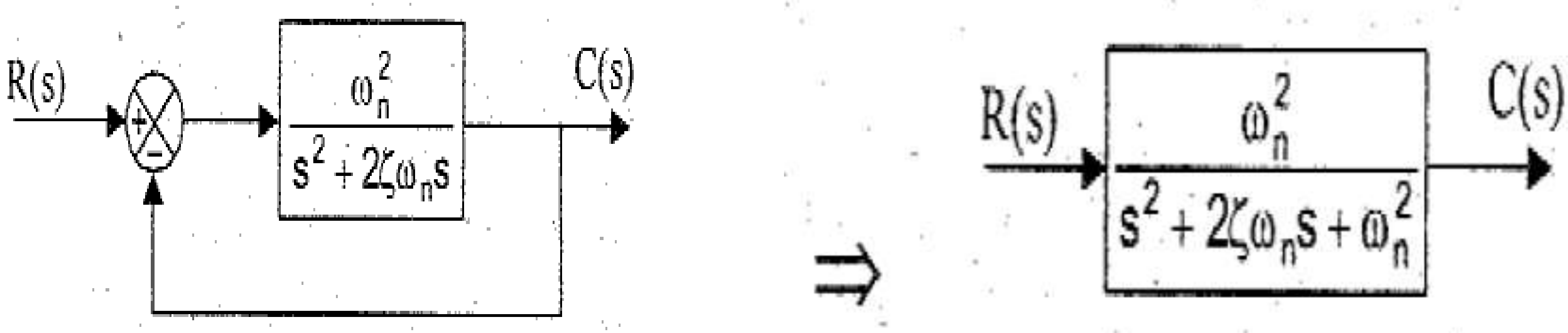
ORDER is Equal to No.of Poles

General Transfer Function Representation of System

First Order System



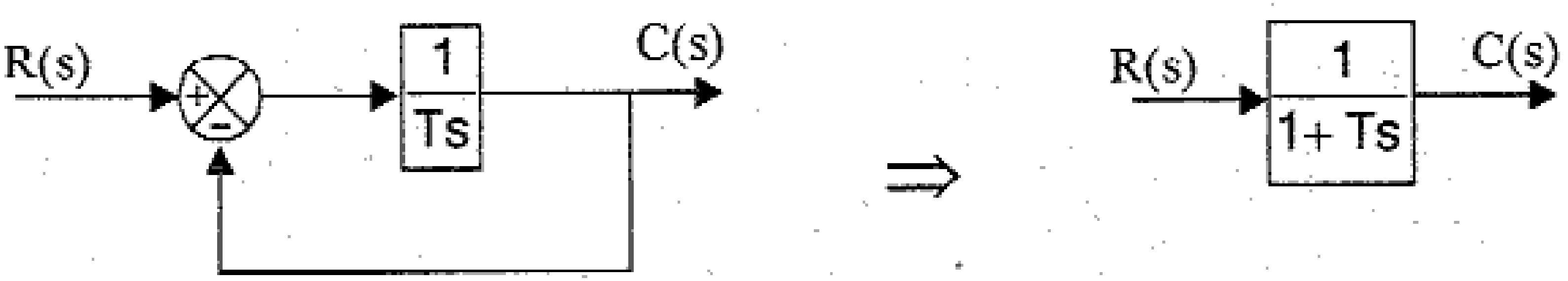
Second Order System



ω_n = Undamped natural frequency, rad/sec.

ζ = Damping ratio.

Response for First Order System for Unit Step Input



The closed loop transfer function of first order system, $\frac{C(s)}{R(s)} = \frac{1}{1+Ts}$

If the input is unit step then, $r(t) = 1$ and $R(s) = \frac{1}{s}$.

$$\therefore \text{The response in s-domain, } C(s) = R(s)\frac{1}{(1+Ts)} = \frac{1}{s}\frac{1}{(1+Ts)} = \frac{1}{sT\left(\frac{1}{T}+s\right)} = \frac{\frac{1}{T}}{s\left(s+\frac{1}{T}\right)}$$

By partial fraction expansion,

$$C(s) = \frac{\frac{1}{T}}{s\left(s+\frac{1}{T}\right)} = \frac{A}{s} + \frac{B}{\left(s+\frac{1}{T}\right)}$$

$$\frac{1}{T} = A\left[s + \frac{1}{T}\right] + Bs$$

① Compare constants

$$\frac{1}{T} = A\frac{1}{T} \Rightarrow A = 1$$

② put $s = -\frac{1}{T}$ $\quad \frac{1}{T} = B\left[-\frac{1}{T}\right]$

$$B = -1$$

The response in time domain is given by,

$$c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s + \frac{1}{T}}\right\} = 1 - e^{-\frac{t}{T}}$$

$$\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$$

When, $t = 0$, $c(t) = 1 - e^{0} = 0$

When, $t = 1T$, $c(t) = 1 - e^{-1} = 0.632$

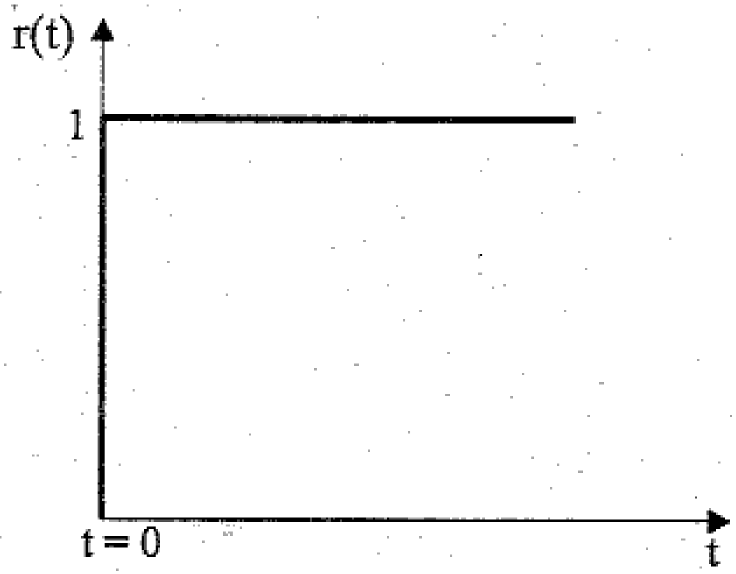
When, $t = 2T$, $c(t) = 1 - e^{-2} = 0.865$

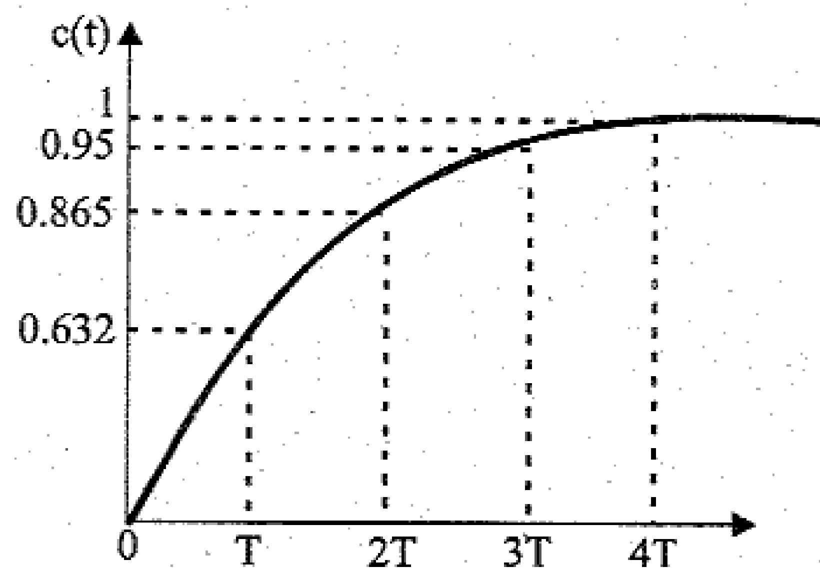
When, $t = 3T$, $c(t) = 1 - e^{-3} = 0.95$

When, $t = 4T$, $c(t) = 1 - e^{-4} = 0.9817$

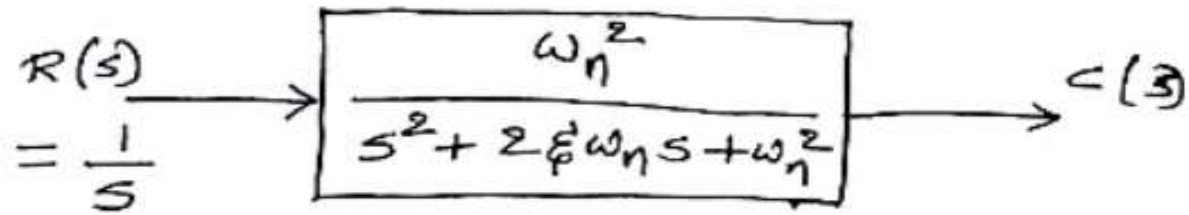
When, $t = 5T$, $c(t) = 1 - e^{-5} = 0.993$

When, $t = \infty$, $c(t) = 1 - e^{-\infty} = 1$


r(t)
1
t = 0
t


c(t)
1
0.95
0.865
0.632
0
T
2T
3T
4T

Second Order System response to UNIT STEP input



$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)}$$

$$c(t) = L^{-1}[C(s)]$$

$$C(s) = \frac{A}{s} + \frac{Bs + C}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$= \frac{A[s^2 + 2\xi\omega_n s + \omega_n^2] + (Bs + C)s}{s(s^2 + 2\xi\omega_n s + \omega_n^2)}$$

$$= \frac{As^2 + 2\xi\omega_n As + A\omega_n^2 + Bs^2 + Cs}{s(s^2 + 2\xi\omega_n s + \omega_n^2)}$$

$$\frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)} = \frac{As^2 + 2\xi\omega_n As + A\omega_n^2 + Bs^2 + Cs}{s(s^2 + 2\xi\omega_n s + \omega_n^2)}$$

$$\omega_n^2 = As^2 + 2\xi\omega_n As + A\omega_n^2 + Bs^2 + Cs$$

compare ω_n^2 coefficients

$$1 = A$$

compare s^2 coefficients

$$0 = A + B \Rightarrow B = -A = -1$$

Compare s coefficients

$$0 = 2\xi\omega_n A + C$$

$$C = -2\xi\omega_n$$

$$C(s) = \frac{1}{s} + \frac{-s-2\xi\omega_n}{s^2+2\xi\omega_n s+\omega_n^2}$$

$$= \frac{1}{s} - \frac{s+2\xi\omega_n}{s^2+2\xi\omega_n s+\omega_n^2}$$

Add and subtract $\xi^2\omega_n^2$ in denominator

$$C(s) = \frac{1}{s} - \frac{s+2\xi\omega_n}{s^2+2\xi\omega_n s+\omega_n^2+\xi^2\omega_n^2-\xi^2\omega_n^2}$$

$$= \frac{1}{s} - \frac{s+2\xi\omega_n}{(s+\xi\omega_n)^2 + \omega_n^2(1-\xi^2)}$$

Let $\omega_n\sqrt{1-\xi^2} = \omega_d$

$$= \frac{1}{s} - \frac{s+2\xi\omega_n}{(s+\xi\omega_n)^2 + \omega_d^2}$$

$$= \frac{1}{s} - \frac{s+\xi\omega_n}{(s+\xi\omega_n)^2 + \omega_d^2} - \frac{\xi\omega_n}{(s+\xi\omega_n)^2 + \omega_d^2}$$

$$= \frac{1}{s} - \frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} - \frac{\xi\omega_n}{\omega_d} \cdot \frac{\omega_d}{(s+\xi\omega_n)^2+\omega_d^2}$$

$$c(t) = L^{-1}[c(s)] = L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}\right]$$

$$- L^{-1}\left[\frac{\omega_d}{(s+\xi\omega_n)^2+\omega_d^2}\right]\frac{\xi\omega_n}{\omega_d}.$$

$$c(t) = 1 - e^{-\xi\omega_n t}\cos\omega_d t - \frac{\xi\omega_n}{\omega_n\sqrt{1-\xi^2}}e^{-\xi\omega_n t}\sin\omega_d t$$

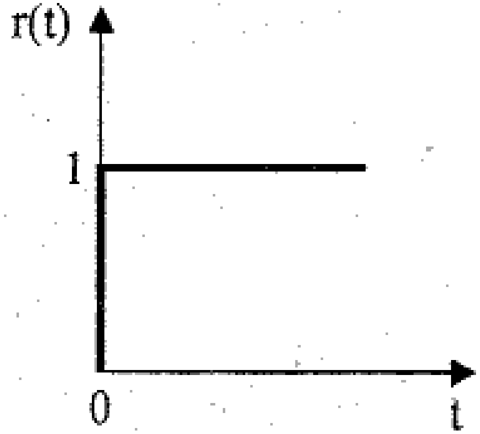
$$= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[\xi \sin\omega_d t + \sqrt{1-\xi^2}\cos\omega_d t\right]$$

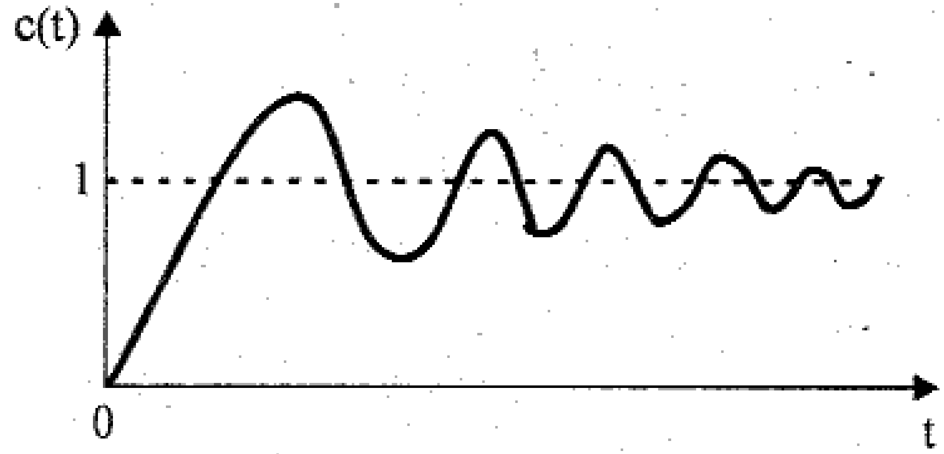
Let $\cos\theta = \xi$ $\sin\theta = \sqrt{1-\xi^2}$

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[\sin(\omega_d t + \theta)\right]$$

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

$$\theta = \tan^{-1}\left[\frac{\sqrt{1-\xi^2}}{\xi}\right]$$

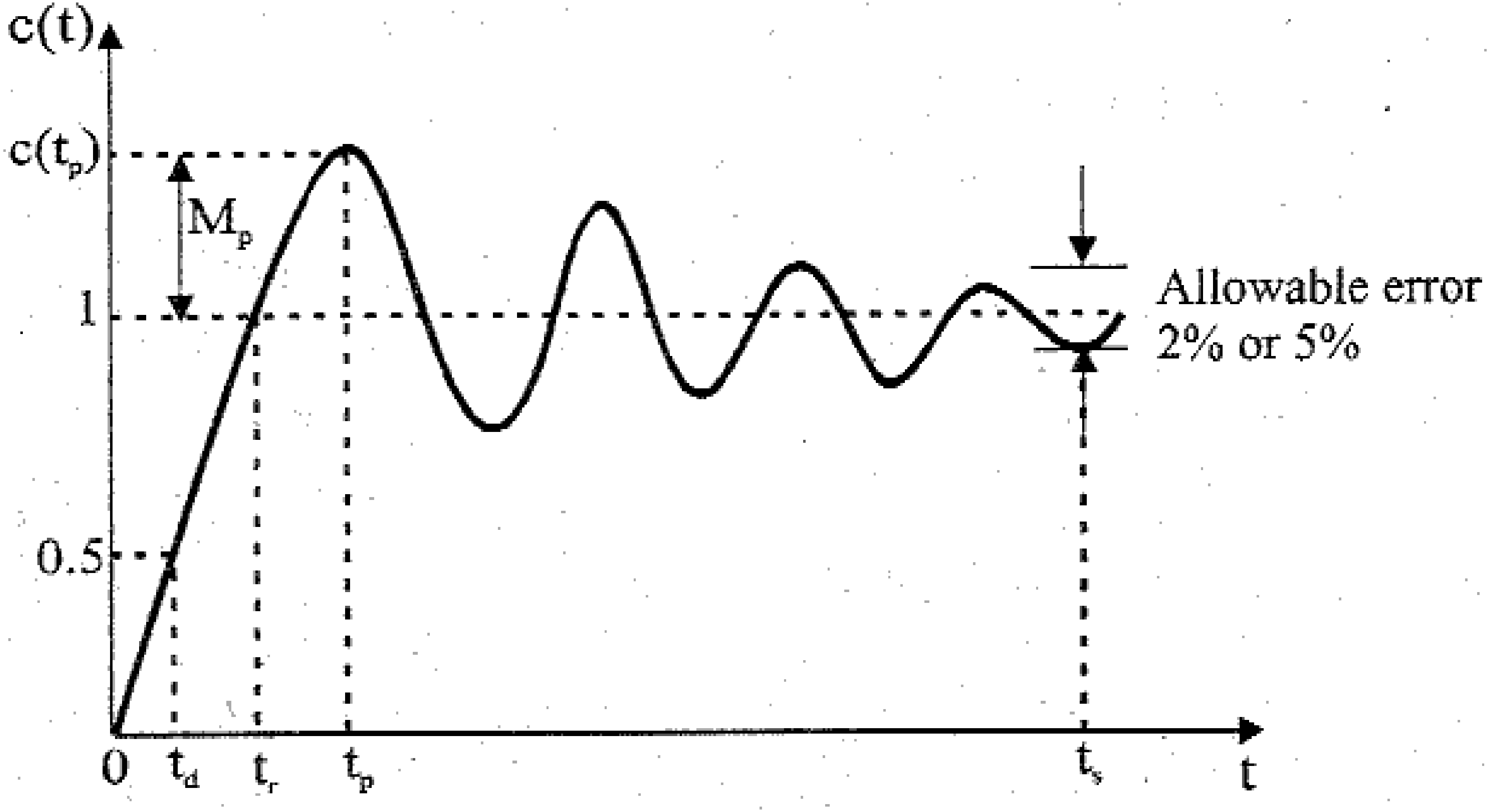




In output response, the system provides oscillations intially and these oscillations are dying with incerease in time. This response is Underdamped System response

$$\xi < 1$$

Output Response of Second Order System for Underdamped System



Time Domain Specifications

Delay time t_d : It is the time taken for the output to reach 50 % of final value for the first time.

Rise time t_r : It is the time taken for the response to raise from 0 to 100 % for the very first time for under damped. (Overdamped 10 % to 90% ,Critical damped 5 % to 95 %)

Peak time tp : It is the time taken for the response to reach the peak value the very first time.

Peak overshoot Mp : It is the ratio of difference between peak and final value to the final value

$$M_p = \frac{c(t_p) - c(\infty)}{c(\infty)}$$

Settling time Ts: It is the time taken for the response to reach and stay within a specified error. It is usually 2 % or 5 %.

Derivation for Rise Time , tr

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)$$

$$\text{At } t = t_r,\ c(t) = c(t_r) = 1$$

$$\therefore c(t_r) = 1 - \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_r + \theta) = 1$$

$$\therefore \frac{-e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_r + \theta) = 0$$

$$-e^{-\zeta\omega_n t_r} \neq 0, \qquad \sin(\omega_d t_r + \theta) = 0$$

$$\sin\phi = 0 \qquad \phi = \pi, 2\pi, 3\pi \ldots,$$

$$\therefore \omega_d t_r + \theta = \pi$$

$$\omega_d t_r = \pi - \theta$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$\therefore \text{ Rise Time, } t_r = \frac{\pi - \theta}{\omega_d}$$

$$\theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$\therefore \text{Rise time, } t_r = \frac{\pi - \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}}{\omega_n\sqrt{1-\zeta^2}} \text{ in sec}$$

Derivation for Peak Time , tp

Differentiating the C(t) with respect to t and equate to zero to get peak time.

$$\frac{d}{dt}c(t)\Big|_{t=t_p} = 0 \qquad c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \theta)$$

$$\frac{d}{dt}c(t) = \frac{-e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}(-\zeta\omega_n)\sin(\omega_d t + \theta) + \left(\frac{-e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\right)\cos(\omega_d t + \theta)\omega_d$$

$$\text{Put, } \omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$\therefore \frac{d}{dt}c(t) = \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}(\zeta\omega_n)\sin(\omega_d t + \theta) - \frac{\omega_n\sqrt{1-\zeta^2}}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\cos(\omega_d t + \theta)$$

$$= \frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\zeta \sin(\omega_d t + \theta) - \sqrt{1-\zeta^2}\ \cos(\omega_d t + \theta)\right]$$

$\cos\theta = \zeta \qquad \sin\theta = \sqrt{1-\zeta^2}$

$$= \frac{\omega_n}{\sqrt{1-\zeta^2}}\ e^{-\zeta\omega_n t}\left[\cos\theta \sin(\omega_d t + \theta) - \sin\theta \cos(\omega_d t + \theta)\right]$$

$$= \frac{\omega_n}{\sqrt{1-\zeta^2}}\ e^{-\zeta\omega_n t}\left[\sin(\omega_d t + \theta)\cos\theta - \cos(\omega_d t + \theta)\sin\theta\right]$$

$$= \frac{\omega_n}{\sqrt{1-\zeta^2}}\ e^{-\zeta\omega_n t}\left[\sin((\omega_d t + \theta) - \theta)\right]$$

$$= \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t)$$

$$\text{at } t = t_p, \ \frac{d}{dt}c(t) = 0$$

$$\therefore \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t_p} \sin(\omega_d t_p) = 0$$

Since, $e^{-\zeta\omega_n t_p} \neq 0$, the term, $\sin(\omega_d t_p) = 0$

When $\phi = 0, \pi, 2\pi, 3\pi, \sin\phi = 0$

$$\therefore \ \omega_d t_p = \pi$$

$$\therefore \text{Peak time, } t_p = \frac{\pi}{\omega_d}$$

$$\omega_d = \omega_n \sqrt{1-\zeta^2}$$

$$\therefore \text{Peak time, } t_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}$$

Derivation for Peak Overshoot

$$\%\text{Peak overshoot, } \%M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100$$

where, $c(t_p)$ = Peak response at $t = t_p$.
$c(\infty)$ = Final steady state value.

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)$$

$$\text{At } t=\infty, \quad c(t) = c(\infty) = 1-\frac{e^{-\infty}}{\sqrt{1-\zeta^2}}\sin(\omega_d t+\theta) = 1-0 = 1$$

$$\text{At } t=t_p, \quad c(t) = c(t_p) = 1-\frac{e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}}\sin(\omega_d t_p+\theta)$$

$$= 1-\frac{e^{-\zeta\omega_n \frac{\pi}{\omega_d}}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d \frac{\pi}{\omega_d}+\theta\right)$$

$$= 1 - \frac{e^{-\zeta\omega_n \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}} \sin(\pi + \theta)}{\sqrt{1-\zeta^2}}$$

$$\sin(\pi + \theta) = -\sin\theta$$

$$= 1 + \frac{e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \sin\theta$$

$$\sin\theta = \sqrt{1-\zeta^2}$$

$$= 1 + \frac{e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \sqrt{1-\zeta^2} = 1 + e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$$

Percentage Peak Overshoot, $\%M_p = \dfrac{c(t_p) - c(\infty)}{c(\infty)} \times 100$

$$= \frac{1 + e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} - 1}{1} \times 100$$

$$= e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100$$

Derivation for Settling Time

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)$$

The response of second order system has two components. They are,

1. Decaying exponential component, $\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}$.
2. Sinusoidal component, $\sin(\omega_d t + \theta)$.

settling time is decided by the exponential component

For 2 % tolerance error band, at $t = t_s$, $\frac{e^{-\zeta\omega_n t_s}}{\sqrt{1-\zeta^2}} = 0.02$

For least values of ζ , $e^{-\zeta\omega_n t_s} = 0.02$.

On taking natural logarithm we get,

$$-\zeta\omega_n t_s = ln(0.02) \quad \Rightarrow \quad -\zeta\omega_n t_s = -4 \quad \Rightarrow \quad t_s = \frac{4}{\zeta\omega_n}$$

$$\therefore \text{ Settling time, } t_s = \frac{1}{\zeta\omega_n} = 4T \qquad \text{(for 2\% error)}$$

For 5% error, $e^{-\zeta\omega_n t_s} = 0.05$

On taking natural logarithm we get,

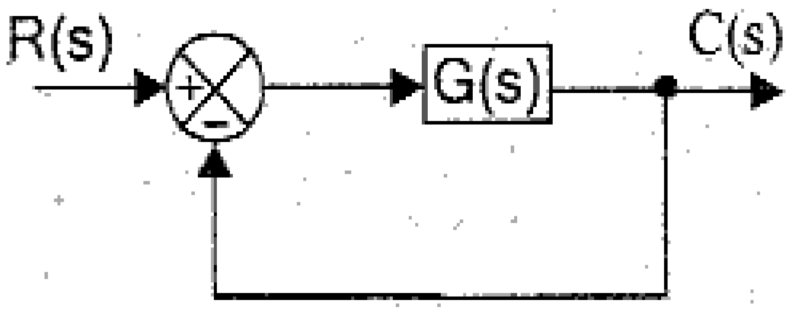
$$-\zeta\omega_n t_s = ln(0.05) \quad \Rightarrow \quad -\zeta\omega_n t_s = -3 \quad \Rightarrow \quad t_s = \frac{3}{\zeta\omega_n}$$

$$\therefore \text{ Settling time, } t_s = \frac{3}{\zeta\omega_n} = 3T \qquad \text{(for 5\% error)}$$

$$\therefore \text{ Settling time, } t_s = \frac{1}{\zeta\omega_n} = 4T \qquad \text{(for 2\% error)}$$

Problem 1

The unity feedback system is characterized by an open loop transfer function G(s) =K/s (s +10). Determine the gain K, so that the system will have a damping ratio of 0.5 for this value of K. Determine peak overshoot and time at peak overshoot for a unit step input.



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} \qquad G(s) = K/s\,(s+10)$$

$$\therefore \frac{C(s)}{R(s)} = \frac{\frac{K}{s(s+10)}}{1+\frac{K}{s(s+10)}} = \frac{K}{s\,(s+10)+K} = \frac{K}{s^2+10s+K}$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{K}{s^2 + 10s + K}$$

$$\omega_n^2 = K \qquad 2\zeta\omega_n = 10 \qquad K = 100$$

$$\therefore \omega_n = \sqrt{K} \qquad \text{Put } \zeta = 0.5 \text{ and } \omega_n = \sqrt{K}$$

$$\therefore 2 \times 0.5 \times \sqrt{K} = 10 \qquad \omega_n = 10 \text{ rad/sec}$$

$$\sqrt{K} = 10$$

Percentage peak overshoot, $\%M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100$

$$= e^{-0.5\pi/\sqrt{1-0.5^2}} \times 100 = 0.163 \times 100 = 16.3\%$$

$$\text{Peak time, } t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{10\sqrt{1-0.5^2}} = 0.363 \text{ sec}$$

Problem 2

A unity feedback control system has an open loop transfer function, G(s) = 10/s(s+2). Find the rise time, percentage overshoot, peak time and settling time for a step input of 12 units.

The closed loop transfer function, $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$

The closed loop transfer function,

Given that, G(s) = 10/s (s+2)

$$\therefore \frac{C(s)}{R(s)} = \frac{\frac{10}{s\,(s+2)}}{1+\frac{10}{s\,(s+2)}} = \frac{10}{s\,(s+2)+10} = \frac{10}{s^2+2s+10}$$

Standard form of Second order transfer function $\left.\right\}$ $\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

On comparing equation (1) & (2) we get,

$$\omega_n^2 = 10 \qquad \therefore\ \omega_n = \sqrt{10} = 3.162 \text{ rad / sec}$$

$$2\zeta\omega_n = 2 \qquad \therefore\ \zeta = \frac{2}{2\omega_n} = \frac{1}{3.162} = 0.316$$

$$\theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} = \tan^{-1}\frac{\sqrt{1-0.316^2}}{0.316} = 1.249 \text{ rad}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 3.162\sqrt{1-0.316^2} = 3 \text{ rad / sec}$$

$$\text{Rise time, } t_r = \frac{\pi - \theta}{\omega_d} = \frac{\pi - 1.249}{3} = 0.63 \text{ sec}$$

Percentage overshoot, $\%M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100 = e^{\frac{-0.316\pi}{\sqrt{1-0.316^2}}} \times 100$

$= 0.3512 \times 100 = 35.12\%$

Peak overshoot $= \frac{35.12}{100} \times 12$ units $= 4.2144$ units

Peak time, $t_p = \frac{\pi}{\omega_d} = \frac{\pi}{3} = 1.047$ sec

Time constant, $T = \frac{1}{\zeta\omega_n} = \frac{1}{0.316 \times 3.162} = 1$ sec

$\therefore$ For 5% error, Settling time, $t_s = 3T = 3$ sec

For 2% error, Settling time, $t_s = 4T = 4$ sec

Rise time, t_r	=	0.63 sec
Percentage overshoot, $\%M_p$	=	35.12%
Peak overshoot	=	4.2144 units, (for a input of 12 units)
Peak time, t_p	=	1.047 sec
Settling time, t_s	=	3 sec for 5% error
	=	4 sec for 2% error

TYPE NUMBER OF CONTROL SYSTEMS

$$G(s)\,H(s) = K\frac{P(s)}{Q(s)} = K\frac{(s+z_1)\,(s+z_2)\,(s+z_3)\,\ldots\ldots\ldots\ldots}{s^N\,(s+p_1)\,(s+p_2)\,(s+p_3)\,\ldots\ldots\ldots}$$

If N = 0, then the system is type – 0 system

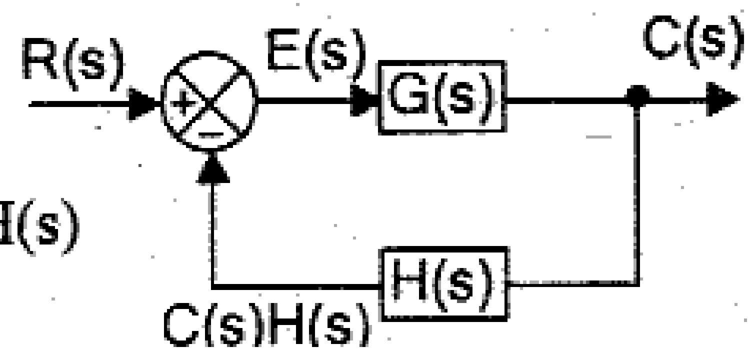
If N = 1, then the system is type – 1 system

If N = 2, then the system is type – 2 system

If N = 3, then the system is type – 3 system and so on.

$$\frac{10(s+2)}{s^2(s+1)} \qquad \frac{20(s+2)}{s(s+1)(s+3)} \qquad \frac{10}{(s+2)(s+3)} \qquad \frac{10}{s^2(s+1)(s+2)}$$

STEADY STATE ERROR



The error signal, $E(s) = R(s) - C(s)\,H(s)$

The output signal, $C(s) = E(s)\,G(s)$

$$E(s) = R(s) - [E(s)\ G(s)]\ H(s)$$

$$E(s) + E(s)\ G(s)\ H(s) = R(s)$$

$$E(s)\ [1 + G(s)\ H(s)] = R(s)$$

$$\therefore E(s) = \frac{R(s)}{1 + G(s)\ H(s)}$$

Using final value theorem,

The steady state error, $e_{ss} = \underset{t \to \infty}{\mathrm{Lt}}\ e(t) = \underset{s \to 0}{\mathrm{Lt}}\ sE(s) = \underset{s \to 0}{\mathrm{Lt}}\ \frac{sR(s)}{1 + G(s)\ H(s)}$

STEADY STATE ERROR WHEN THE INPUT IS UNIT STEP SIGNAL

When the input is unit step, $R(s) = 1/s$

$$\text{Steady state error, } e_{ss} = \underset{s\to 0}{Lt} \frac{sR(s)}{1+G(s)H(s)}$$

$$\therefore e_{ss} = \underset{s\to 0}{Lt} \frac{s\frac{1}{s}}{1+G(s)H(s)} = \underset{s\to 0}{Lt} \frac{1}{1+G(s)H(s)} = \frac{1}{1+\underset{s\to 0}{Lt} G(s)H(s)} = \frac{1}{1+K_p}$$

$$\text{where, } K_p = \underset{s\to 0}{Lt} G(s)H(s)$$

The constant K_p is called ***positional error constant***.

Type-0 system

$$K_p = \underset{s\to 0}{Lt}\, G(s)\,H(s) = \underset{s\to 0}{Lt}\, K\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{(s+p_1)\,(s+p_2)\,(s+p_3)......}$$

$$K\frac{z_1.z_2.z_3......}{p_1.p_2.p_3......} = \text{constant}$$

$$\therefore e_{ss} = \frac{1}{1+K_p} = \text{constant}$$

Hence in type-0 systems when the input is unit step there will be a constant steady state error.

Type-1 system

$$K_p = \underset{s\to 0}{Lt}\, G(s)\, H(s) = \underset{s\to 0}{Lt}\, K \frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{s\,(s+p_1)\,(s+p_2)\,(s+p_3)......} = \infty$$

$$\therefore e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0$$

In systems with type number 1 and above, for unit step input the value of K_p is infinity and so the steady state error is zero.

STEADY STATE ERROR WHEN THE INPUT IS UNIT RAMP SIGNAL

$$\text{Steady state error, } e_{ss} = \underset{s\to 0}{Lt} \frac{sR(s)}{1+G(s)\,H(s)}$$

$$\text{When the input is unit ramp, } R(s) = \frac{1}{s^2}$$

$$\therefore e_{ss} = \underset{s\to 0}{Lt} \frac{s\frac{1}{s^2}}{1+G(s)\,H(s)} = \underset{s\to 0}{Lt} \frac{1}{s+sG(s)\,H(s)}$$

$$\frac{1}{\underset{s\to 0}{Lt}\ sG(s)\,H(s)} = \frac{1}{K_v}$$

$$\text{where, } K_v = \underset{s\to 0}{Lt}\ s\,G(s)\,H(s)$$

The constant K_v is called ***velocity error constant.***

Type-0 system

$$K_v = \underset{s \to 0}{Lt}\, sG(s)\,H(s) = \underset{s \to 0}{Lt}\; sK\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{(s+p_1)\,(s+p_2)\,(s+p_3)......} = 0$$

$$\therefore e_{ss} = 1/K_v = 1/0 = \infty$$

Hence in type-0 systems when the input is unit ramp, the steady state error is infinity.

Type-1 system

$$K_v = \underset{s \to 0}{Lt}\, sG(s)\,H(s) = \underset{s \to 0}{Lt}\; sK\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{s\,(s+p_1)\,(s+p_2)\,(s+p_3)......} = K\frac{z_1.z_2.z_3......}{p_1.p_2.p_3......} = \text{constant}$$

$$\therefore e_{ss} = 1/K_v = \text{constant}$$

Hence in type-1 systems when the input is unit ramp there will be a constant steady state error.

Type-2 system

$$K_v = \underset{s \to 0}{Lt}\, sG(s)\,H(s) = \underset{s \to 0}{Lt}\; sK\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{s^2\,(s+p_1)\,(s+p_2)\,(s+p_3)......} = \infty$$

$$\therefore e_{ss} = 1/K_v = 1/\infty = 0$$

In systems with type number 2 and above, for unit ramp input, the value of K_v is infinity so the steady state error is zero.

STEADY STATE ERROR WHEN THE INPUT IS UNIT PARABOLIC SIGNAL

Steady state error, $e_{ss} = \underset{s\to 0}{Lt} \dfrac{sR(s)}{1+G(s)\,H(s)}$

When the input is unit parabola, $R(s) = \dfrac{1}{s^3}$

$$\therefore e_{ss} = \underset{s\to 0}{Lt} \frac{s\dfrac{1}{s^3}}{1+G(s)\,H(s)} = \underset{s\to 0}{Lt} \frac{1}{s^2+s^2\,G(s)\,H(s)} = \frac{1}{\underset{s\to 0}{Lt}\ s^2G(s)\,H(s)} = \frac{1}{K_a}$$

where, $K_a = \underset{s\to 0}{Lt}\ s^2G(s)\,H(s)$

The constant K_a is called ***acceleration error constant.***

Type-0 system

$$K_a = \underset{s\to 0}{Lt}\, s^2 G(s)\,H(s) = \underset{s\to 0}{Lt}\; s^2 K \frac{(s+z_1)\,(s+z_2)\,(s+z_3)\ldots\ldots}{(s+p_1)\,(s+p_2)\,(s+p_3)\ldots\ldots} = 0$$

$$\therefore e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty$$

Hence in type-0 systems for unit parabolic input, the steady state error is infinity.

Type-1 system

$$K_a = \underset{s\to 0}{Lt}\, s^2 G(s)\,H(s) = \underset{s\to 0}{Lt}\; s^2 K \frac{(s+z_1)\,(s+z_2)\,(s+z_3)\ldots\ldots}{s\,(s+p_1)\,(s+p_2)\,(s+p_3)\ldots\ldots} = 0$$

$$\therefore e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty$$

Hence in type-1 systems for unit parabolic input, the steady state error is infinity.

Type-2 system

$$K_a = \underset{s\to 0}{Lt}\, s^2 G(s)\, H(s) = \underset{s\to 0}{Lt}\; s^2 K \frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{s^2(s+p_1)\,(s+p_2)\,(s+p_3)......} = K\frac{z_1.z_2.z_3......}{p_1.p_2.p_3.....} = \text{constant}$$

$$\therefore e_{ss} = \frac{1}{K_a} = \text{constant}$$

Hence in type-2 system when the input is unit parabolic signal there will be a constant steady state error.

Type-3 system

$$K_a = \underset{s\to 0}{Lt}\, s^2 G(s)\, H(s) = \underset{s\to 0}{Lt}\; s^2 K \frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{s^3\,(s+p_1)\,(s+p_2)\,(s+p_3)......} = \infty$$

$$\therefore e_{ss} = \frac{1}{K_a} = \frac{1}{\infty} = 0$$

In systems with type number 3 and above for unit parabolic input the value of K_a is infinity and so the steady state error is zero.

$$K_p = \underset{s \to 0}{Lt}\ G(s)\,H(s)$$

$$K_v = \underset{s \to 0}{Lt}\ s\,G(s)\,H(s)$$

$$K_a = \underset{s \to 0}{Lt}\ s^2 G(s)\,H(s)$$

TABLE-2.2 : Static Error Constant for Various Type Number of Systems

Error Constant	Type number of system			
	0	1	2	3
K_p	constant	∞	∞	∞
K_v	0	constant	∞	∞
K_a	0	0	constant	∞

As Type Number Increases Steady State Error Decreases

TABLE-2.3 : Steady State Error for Various Types of Inputs

Input Signal	Type number of system			
	0	1	2	3
Unit Step	$\frac{1}{1+K_p}$	0	0	0
Unit Ramp	∞	$\frac{1}{K_v}$	0	0
Unit Parabolic	∞	∞	$\frac{1}{K_a}$	0

Problem 3

For a unity feedback control system the open loop transfer function, $G(s) = \frac{10(s+2)}{s^2(s+1)}$. Find

a) the position, velocity and acceleration error constants,

For a unity feedback system , H(s)=1

Position error constant, $K_p = \underset{s \to 0}{Lt}\, G(s)H(s) = \underset{s \to 0}{Lt}\, G(s)$

$$= \underset{s \to 0}{Lt} \frac{10(s+2)}{s^2(s+1)} = \infty$$

Velocity error constant, $K_v = \underset{s \to 0}{Lt}\, s\, G(s)H(s) = \underset{s \to 0}{Lt}\, s\, G(s) = \underset{s \to 0}{Lt}\, s \frac{10(s+2)}{s^2(s+1)} = \infty$

Acceleration error constant, $K_a = \underset{s \to 0}{Lt}\, s^2 G(s)H(s) = \underset{s \to 0}{Lt}\, s^2 G(s)$

$$= \underset{s \to 0}{Lt}\, s^2 \frac{10(s+2)}{s^2(s+1)} = \frac{10 \times 2}{1} = 20$$

$$K_p = \lim_{s \to 0} G(s)\,H(s)$$

$$K_v = \lim_{s \to 0} s\,G(s)\,H(s)$$

$$K_a = \lim_{s \to 0} s^2 G(s)\,H(s)$$

Input Signal	Type number of system			
	0	1	2	3
Unit Step	$\frac{1}{1+K_p}$	0	0	0
Unit Ramp	∞	$\frac{1}{K_v}$	0	0
Unit Parabolic	∞	∞	$\frac{1}{K_a}$	0

Problem 4

constant steady state error and calculate their values.

a) $G(s) = \frac{20(s+2)}{s(s+1)(s+3)}$; b) $G(s) = \frac{10}{(s+2)(s+3)}$; c) $G(s) = \frac{10}{s^2(s+1)(s+2)}$

a) $G(s) = \dfrac{20(s+2)}{s(s+1)(s+3)}$

Let us assume unity feedback system, $\therefore H(s)=1$

The open loop system has a pole at origin. Hence it is a type-1 system. In systems with type number-1, the velocity (ramp) input will give a constant steady state error.

The steady state error with unit velocity input, $e_{ss} = \dfrac{1}{K_v}$

Velocity error constant, $K_v = \underset{s\to 0}{Lt}\ s\,G(s)\,H(s) = \underset{s\to 0}{Lt}\ s\,G(s)$

$$= \underset{s\to 0}{Lt}\ s\,\frac{20(s+2)}{s(s+1)(s+3)} = \frac{20\times 2}{1\times 3} = \frac{40}{3}$$

Steady state error, $e_{SS} = \dfrac{1}{K_v} = \dfrac{3}{40} = 0.075$

b) $G(s) = \frac{10}{(s+2)(s+3)}$

Let us assume unity feedback system, $\therefore H(s)=1$.

The open loop system has no pole at origin. Hence it is a type-0 system. In systems with type number-0, the step input will give a constant steady state error.

The steady state error with unit step input, $e_{ss} = \frac{1}{1+K_p}$

Position error constant, $K_p = \underset{s \to 0}{Lt}\ G(s)H(s) = \underset{s \to 0}{Lt}\ G(s) = \underset{s \to 0}{Lt} \frac{10}{(s+2)(s+3)} = \frac{10}{2 \times 3} = \frac{5}{3}$

Steady state error, $e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\frac{5}{3}} = \frac{3}{3+5} = \frac{3}{8} = 0.375$

c) $G(s) = \frac{10}{s^2(s+1)(s+2)}$

Let us assume unity feedback system, $\therefore H(s)=1$.

The open loop system has two poles at origin. Hence it is a type-2 system. In systems with type number-2, the acceleration (parabolic) input will give a constant steady state error.

The steady state error with unit acceleration input, $e_{ss} = \frac{1}{K_a}$

Acceleration error constant, $K_a = \underset{s\to 0}{Lt}\ s^2 G(s)H(s) = \underset{s\to 0}{Lt}\ s^2 G(s) = \underset{s\to 0}{Lt}\ s^2 \frac{10}{s^2(s+1)(s+2)} = \frac{10}{1\times 2} = 5$

Steady state error, $e_{ss} = \frac{1}{K_a} = \frac{1}{5} = 0.2$

Response for Undamped, Critically damped and Overdamed systems for Unit Step Input

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

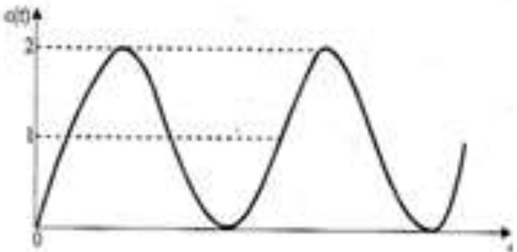
where, ω_n = Undamped natural frequency, rad/sec.

ζ = Damping ratio.

Case 1	:	Undamped system,	$\zeta = 0$
Case 2	:	Under damped system,	$0 < \zeta < 1$
Case 3	:	Critically damped system,	$\zeta = 1$
Case 4	:	Over damped system,	$\zeta > 1$

RESPONSE OF UNDAMPED SECOND ORDER SYSTEM FOR UNIT STEP INPUT

$$C(s) = R(s)\,\frac{\omega_n^2}{s^2+\omega_n^2} = \frac{1}{s}\,\frac{\omega_n^2}{s^2+\omega_n^2}$$



By partial fraction expansion,

$$C(s) = \frac{\omega_n^2}{s\,(s^2+\omega_n^2)} = \frac{A}{s} + \frac{B}{s^2+\omega_n^2}$$

$$\omega_n^2 = A(s^2+\omega_n^2) + Bs$$

1) compare ω_n^2 coeff

2) $s^2 = -\omega_n^2$

$$C(s) = \frac{A}{s} + \frac{B}{s^2+\omega_n^2} = \frac{1}{s} - \frac{s}{s^2+\omega_n^2}$$

$$c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s}{s^2+\omega_n^2}\right\} = 1 - \cos\omega_n t$$

RESPONSE OF CRITICALLY DAMPED SECOND ORDER SYSTEM FOR UNIT STEP INPUT

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

For critical damping $\zeta = 1$.

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\omega_n)^2}$$

$$C(s) = R(s)\frac{\omega_n^2}{(s+\omega_n)^2} = \frac{1}{s}\,\frac{\omega_n^2}{(s+\omega_n)^2} = \frac{\omega_n^2}{s\,(s+\omega_n)^2}$$

By partial fraction expansion, we can write,

$$C(s) = \frac{\omega_n^2}{s\,(s+\omega_n)^2} = \frac{A}{s} + \frac{B}{(s+\omega_n)^2} + \frac{C}{s+\omega_n}$$

$$\omega_n^2 = A(s+\omega_n)^2 + Bs(s+\omega_n) + Cs$$

1) compare ω_n^2 coeff

$$1 = A$$

2) put $s = -\omega_n$

$$\omega_n^2 = C(-\omega_n) \qquad C = -\omega_n$$

3) compare s^2 coeff

$$0 = A + B$$

$$B = -A = -1$$

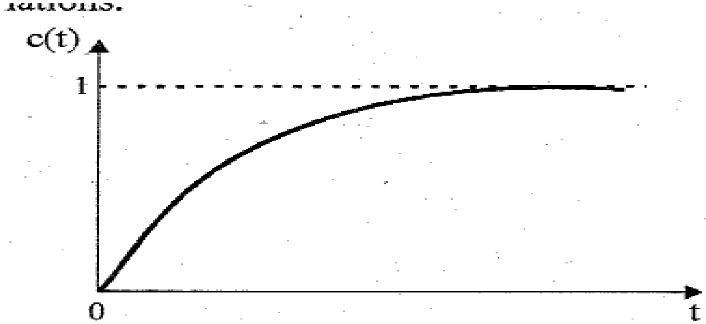
$$\therefore C(s) = \frac{A}{s} + \frac{B}{(s+\omega_n)^2} + \frac{C}{s+\omega_n} = \frac{1}{s} - \frac{\omega_n}{(s+\omega_n)^2} - \frac{1}{s+\omega_n}$$

The response in time domain,

$$c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{\omega_n}{(s+\omega_n)^2} - \frac{1}{s+\omega_n}\right\}$$

$$c(t) = 1 - \omega_n t\, e^{-\omega_n t} - e^{-\omega_n t}$$

$$c(t) = 1 - e^{-\omega_n t}(1+\omega_n t)$$

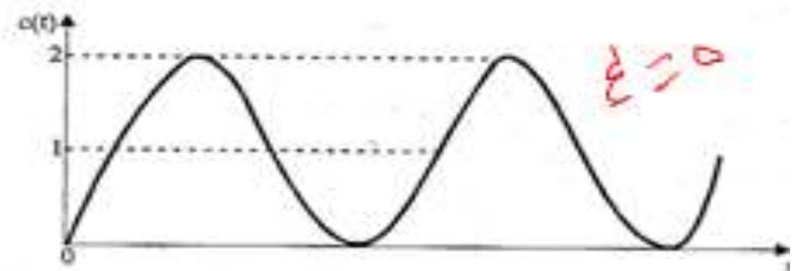


$$\mathcal{L}\{1\} = \frac{1}{s}$$

$$\mathcal{L}\{te^{-at}\} = \frac{1}{(s+a)^2}$$

$$\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$$

Marginally Stable System

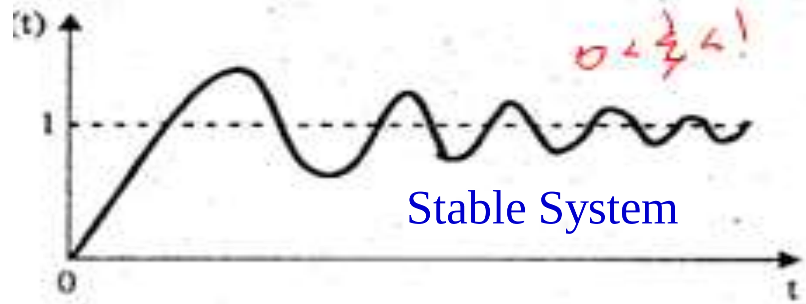


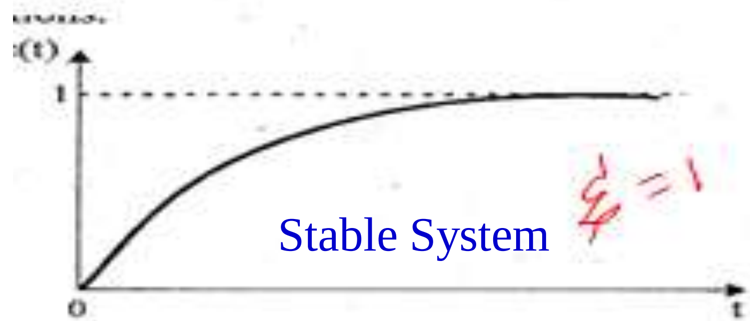
Case 1 : Undamped system, $\zeta = 0$

Case 2 : Under damped system, $0 < \zeta < 1$

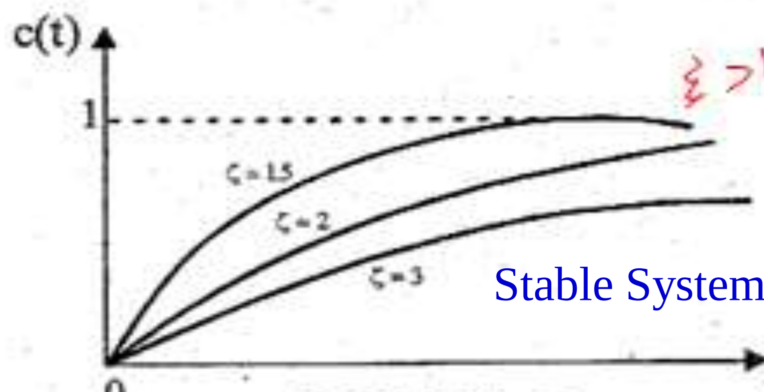
Case 3 : Critically damped system, $\zeta = 1$

Case 4 : Over damped system, $\zeta > 1$





Oscillations Decreasing as ζ increases



Poles of Second Order System for Undamped, Under damped, Critically damped and Over damped systems

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

where, ω_n = Undamped natural frequency, rad/sec.

ζ = Damping ratio.

The characteristics equation of the second order system is,

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

It is a quadratic equation and the roots of this equation is given by,

$$s_1, s_2 = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2} = \frac{-2\zeta\omega_n \pm \sqrt{4\omega_n^2(\zeta^2 - 1)}}{2}$$

$$= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

When $\zeta = 0,\ s_1, s_2 = \pm j\omega_n$; $\begin{cases}\text{roots are purely imaginary} \\ \text{and the system is undamped}\end{cases}$

When $\zeta = 1,\ s_1, s_2 = -\omega_n$; $\begin{cases}\text{roots are real and equal and} \\ \text{the system is critically damped}\end{cases}$

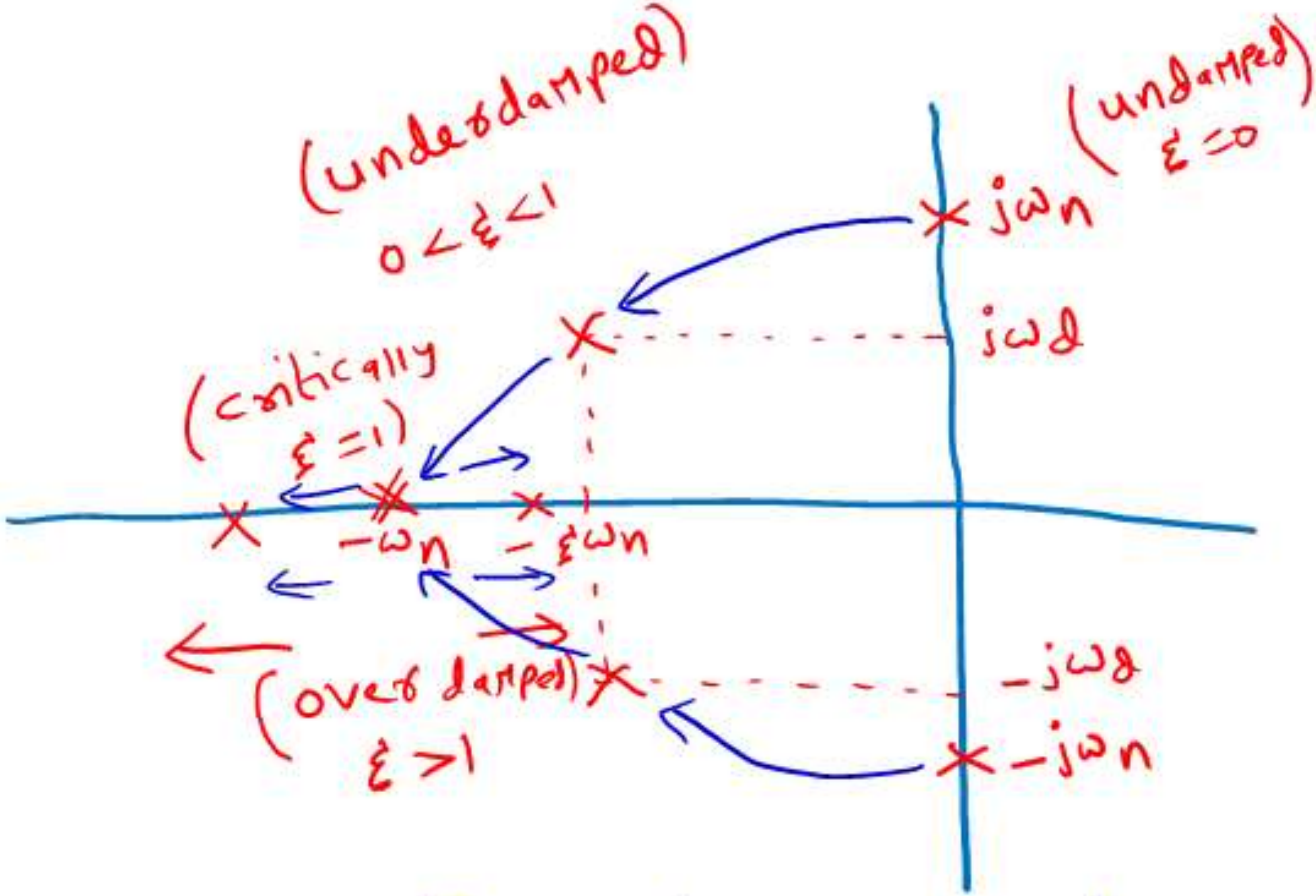
When $\zeta > 1,\ s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$; $\begin{cases}\text{roots are real and unequal and} \\ \text{the system is overdamped}\end{cases}$

When $0 < \zeta < 1,\ s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -\zeta\omega_n \pm \omega_n\sqrt{(-1)\,(1-\zeta^2)}$

$$= -\zeta\omega_n \pm \omega_n\sqrt{-1}\ \sqrt{1-\zeta^2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

$= -\zeta\omega_n \pm j\omega_d$; $\begin{cases}\text{roots are complex conjugate} \\ \text{the system is underdamped}\end{cases}$

where, $\omega_d = \omega_n\sqrt{1-\zeta^2}$



As ξ↑ → Poles Moves towards Left half of s plane.

① As Type number increases, the steady state error decreases and steady state response improved.

② As ζ increases, the oscillations decreases and stability increases and transient response improved.

③ A zero added to system stability increases and pole is added to system stability decreases.

Effect of P, PI, PD, PID Controllers

1) **Order of system** :-

Transfer function

$$T = \frac{C(s)}{R(s)} = \frac{a_0 s^m + a_1 s^{m-1} + \dots a_n}{s^n + b_1 s^{n-1} + \dots + b_n}$$

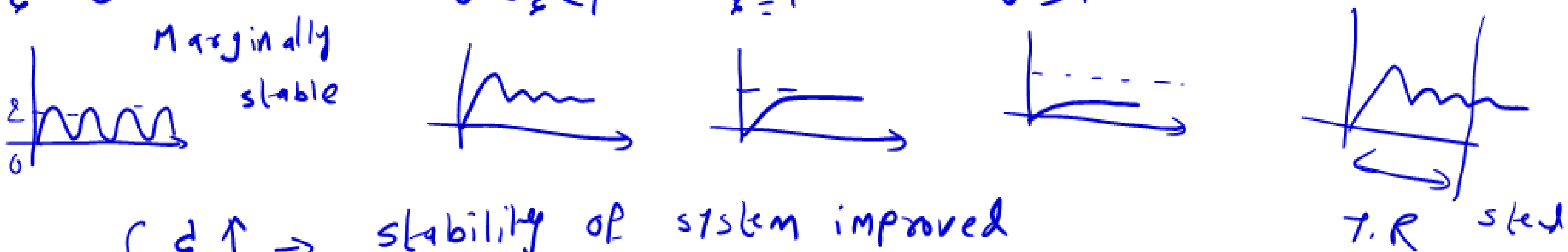
Highest power of s in denominator

2) **Type of system** :-

$$G(s)H(s) = \frac{k(s+z_1)(s+z_2)\dots}{s^N (s+p_1)(s+p_2)\dots}$$

N - type number (poles at origin)

3) $\xi = 0$ $\quad 0 < \xi < 1$ $\quad \xi = 1$ $\quad \xi > 1$



$\xi \uparrow$ → stability of system improved

→ oscillations ↓

→ transient response is improved

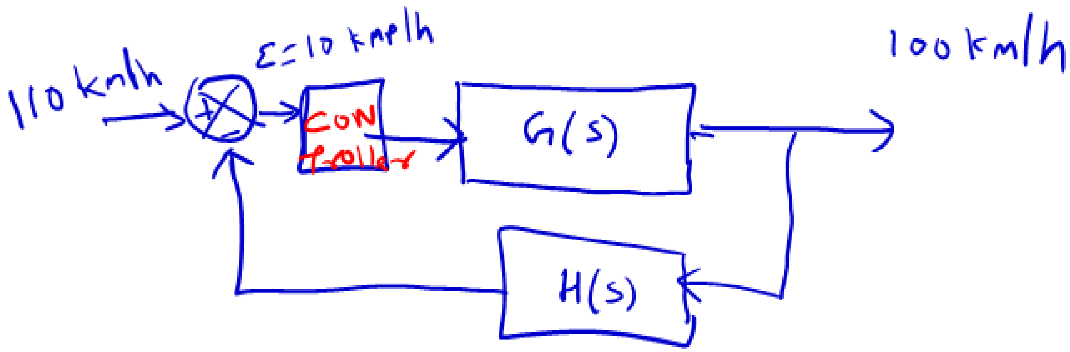
4) As Type no ↑ → e_{ss} ↓ → steady state response is improved.

5) A zero is added to system → stability of system increases

A pole is " " " → stability of system decreases.

Controllers (P, PI, PD, PID)

Effects of P, PI, PD, PID



1) P controller

$u(t) \propto e(t)$

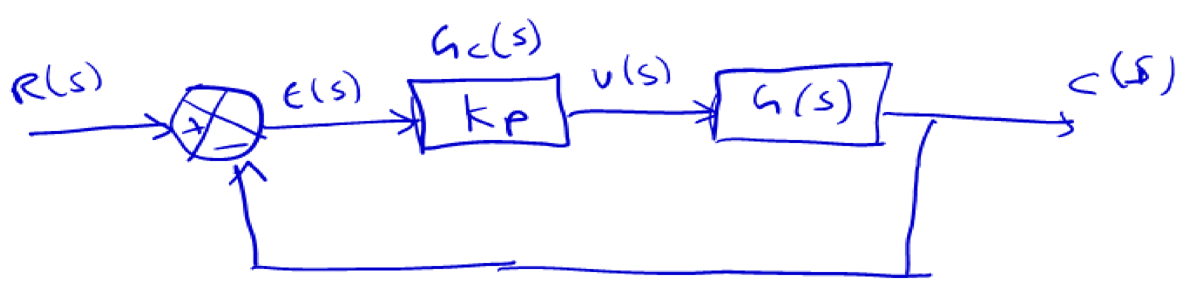
↑ controller o/p

$u(t) = k_p \cdot e(t)$ { k_p = proportionality constant }

Apply L.T on both sides

$U(s) = k_p E(s)$

$$G_c(s) = \frac{U(s)}{E(s)} = k_p$$



observations

1) Gain increases
2) Steady state error constant

2) Proportional Integrator (PI)

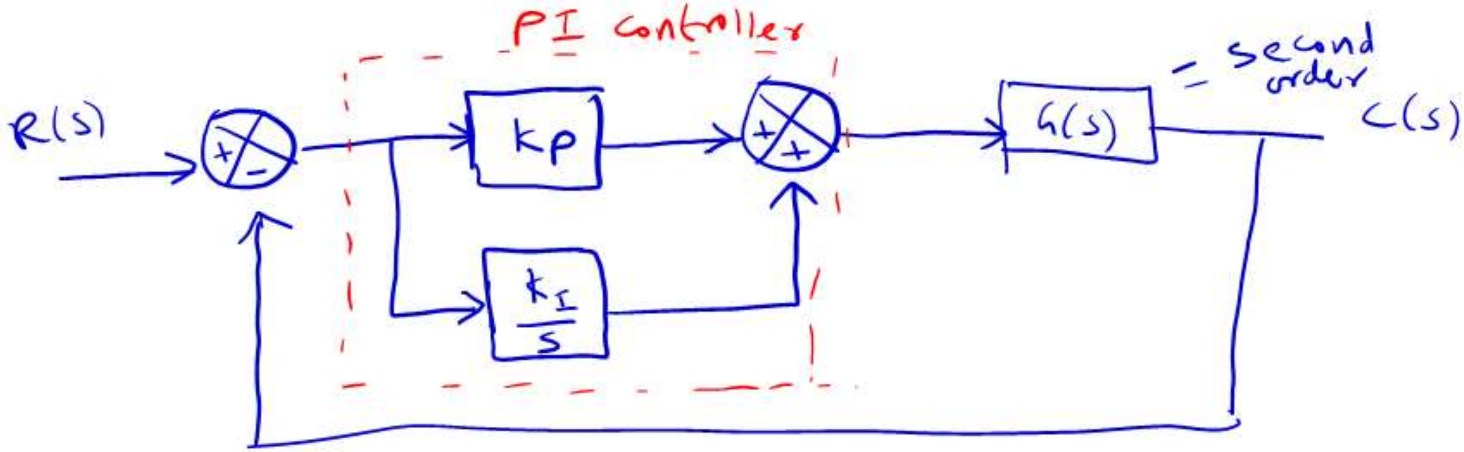
$$u(t) \propto e(t) + \int e(t)\, dt$$

$$u(t) = k_p e(t) + k_I \int e(t)\, dt$$

Apply L.T on both sides

$$U(s) = k_p E(s) + k_I \frac{E(s)}{s}$$

$$G_c(s) = \frac{U(s)}{E(s)} = k_p + \frac{k_I}{s} = \frac{k_p s + k_I}{s}$$



With out controller $G(s)\,H(s) = G(s)\,H(s)$

$$= \frac{\omega_n^2}{s(s+2\xi\omega_n)}$$

with controller $G_c(s)\,G(s)\,H(s) = \left(k_p + \frac{k_I}{s}\right)\frac{\omega_n^2}{s(s+2\xi\omega_n)}$

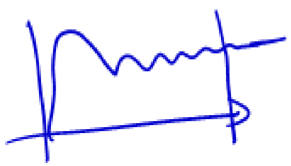
$$= \left(\frac{k_p s + k_I}{s}\right)$$

observations

1) Zero and Pole are added

Zero compensates Pole, stability constant

2) As Type no ↑ ; steady state Error ↓



steady state response improve

(Improved damping)

PD Controller

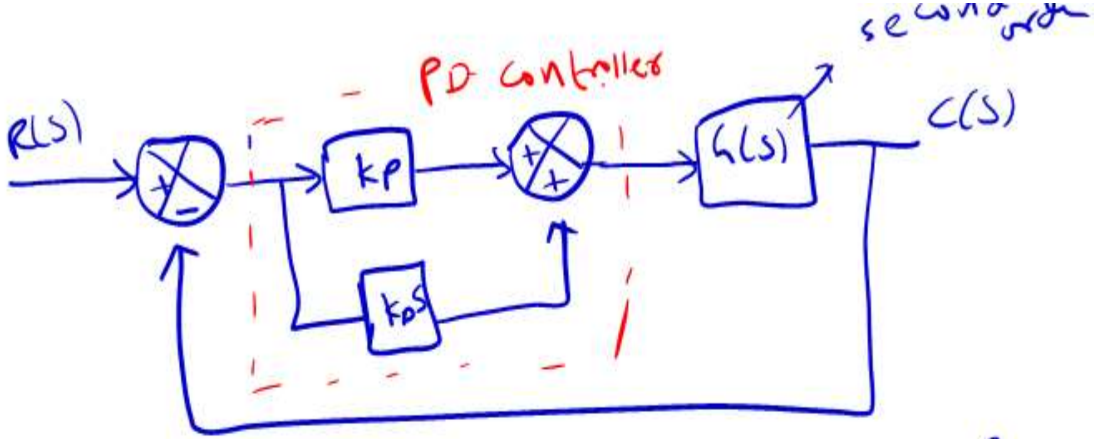
$$u(t) \propto e(t) + \frac{d}{dt} e(t)$$

$$u(t) = K_p e(t) + K_D \frac{d}{dt} e(t)$$

Apply L.T on both sides

$$U(s) = K_p E(s) + K_D s \cdot E(s)$$

$$G_c(s) = \frac{U(s)}{E(s)} = K_p + K_D s.$$



without controller $G(s)H(s) = \frac{\omega_n^2}{s(s+2\xi\omega_n)}$

with controller $G_c(s)\,G(s)H(s) = (K_p + K_D s)\,\frac{\omega_n^2}{s(s+2\xi\omega_n)}$

Observation

① Zero is added; stability ↑; ξ ↑; oscillations ↓

T_p ↓, T_r ↓, M_p ↓ . Transient response improves.

② Type no is constant → e_{ss} constant → steady state response is constant

4) PID controller

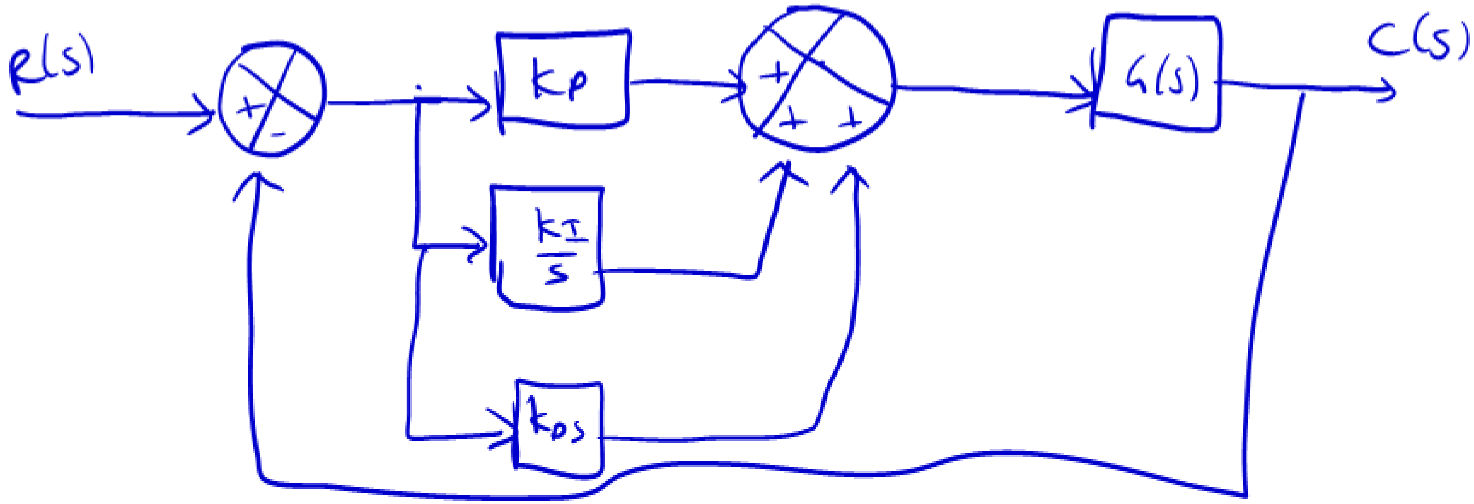
$$u(t) \propto e(t) + \int e(t)\,dt + \frac{d}{dt} e(t)$$

$$u(t) = k_P\, e(t) + k_I \int e(t)\,dt + k_D \frac{d}{dt} e(t)$$

Apply L.T on both sides

$$U(s) = k_P E(s) + \frac{k_I}{s} E(s) + k_D s\, E(s)$$

$$G_c(s) = \frac{U(s)}{E(s)} = \left(k_P + \frac{k_I}{s} + k_D s\right)$$



without controller $G(s)H(s) = \dfrac{\omega_n^2}{s(s+2\xi\omega_n)}$

with controller $\left(K_P + \dfrac{K_I}{s} + K_D s\right) \dfrac{\omega_n^2}{s(s+2\xi\omega_n)}$

$$\underbrace{\left(K_P + \frac{K_I}{s} + K_D s\right)}_{\left(\frac{K_P s + K_I + K_D s^2}{s}\right)}$$

Observations. Two zeros ; one pole adding

one zero compensate one pole

2nd zero, ↑ stability, & ↑ → oscillation ↓

T.R improve

2) S → Type no ↑, ess ↓, steady state regime

improve.

1. P Controller (Proportional Controller)

Gain Increases, Seady State Error Constant

2. PI Controller (Proportional Integrator Controller)

Steady State Response Improved

3. PD Controller (Proportional Derivative Controller)

Transient Response Improved (Rise time, Peak time and peak overshoot decreases)

4. PID Controller (Proportional Integrator and Derivative Controller)

Transient and Steady state response both are improved